Optimal energy decay rate for the damped wave equations on $\mathbb{T}^2$

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The damped wave equation

Damped wave equation on a compact manifold M without boundary:

$$\begin{cases} \partial_t^2 u - \Delta_g u + a(x)\partial_t u = 0, & (t,x) \in \mathbb{R}_+ \times M, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1(M) \times L^2(M) \end{cases}$$

$u(t,x) \in \mathbb{R}$, $a \in C(M)$ $a(x) \geq 0$ and $\int_M a > 0$.

- Energy:

$$E[u](t) := \frac{1}{2} \int_M |\partial_t u|^2 + |\nabla_g u|^2 dx.$$

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- Dissipation:

$$\begin{aligned} \frac{d}{dt}E[u](t) &= \int_M \partial_t u \partial_t^2 u + \nabla_g \partial_t u \cdot \nabla_g u dx \\ &= \int_M \partial_t u(\partial_t^2 u - \Delta_g u) dx \\ &= -\int_M a(x)|\partial_t u|^2 dx \leq 0. \end{aligned}$$

- Main question: Uniform energy decay rate in terms of the size of initial data?
- It turns out that the rate is related to the underlying manifold (global geometry) and the vanishing property of $a(x)$ (local geometry).

Well-posedness as contraction semigroup

- The damped wave equation can be written as

$$\partial_t U(t) = \mathcal{A} U(t)$$

on $\mathcal{H} := \dot{H}^1(M) \times L^2(M)$, where $U(t) = (u(t), \partial_t u(t))^t$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta_g & -a(x) \end{pmatrix}.$$

- $\mathcal{A}$ is m-dissipative $\Longrightarrow$ $(e^{t\mathcal{A}})_{t\geq 0}$ is a contraction semigroup.

$$E[u(t)] = \frac{1}{2}\|e^{t\mathcal{A}}(u_0, u_1)\|^2_{\mathcal{H}}.$$

- By the unique continuation theorem of the Laplace operator, we have $\text{Spec}(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Indeed, if $i\lambda \in \text{Spec}(\mathcal{A})$ for some $\lambda \in \mathbb{R}$ and $(u, v)^t$ is the corresponding eigenvector (non-constant), then

$$-\Delta u - \lambda^2 u + ia(x)\lambda u = 0.$$

W.L.O.G., assume that $\lambda \neq 0$. Then

$$\int_M a(x)|u|^2 dx = 0, \implies u \equiv 0 \text{ on } \omega := \{a > 0\}.$$

Thus $-\Delta u = \lambda^2 u$, a.e. Thus UCP $\implies$ $u \equiv 0$.

Semigroup decay=Resolvent estimate

Resolvent: $R(z) := (z - \mathcal{A})^{-1}$, $z \in \mathbb{C}$.

- Gearhart's theorem: For $\mathcal{A}$ such that $\mathrm{Spec}(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, there is an equivalence between
  - (a) $\|e^{t\mathcal{A}}(u_0, v_0)\|_{\mathcal{H}} \leq e^{-ct}\|(u_0, v_0)\|_{\mathcal{H}}$
  - (b) $\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C$, for all $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$.
- Borichev-Tomilov's theorem: For $\mathcal{A}$ such that $\mathrm{Spec}(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, there is an equivalence between
  - (a) $\|e^{t\mathcal{A}}(u_0, v_0)\|_{\mathcal{H}} \leq Ct^{-\frac{1}{\alpha}}\|(u_0, v_0)\|_{H^2 \times H^1}$, for $t \geq 1$;
  - (b) $\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C(1 + |\lambda|)^{\alpha}$, for $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$.
- The matix resolvent estimate

$$\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C(1 + |\lambda|)^{\alpha}$$

is equivalent to the scalar resolvent estimate (writing $h = \lambda^{-1} \ll 1$)

$$\|P_{h,a}^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-\alpha-1}, \tag{1}$$

where $P_{h,a} = -h^2\Delta - 1 + iha(x)$. This allows us to apply tools in semiclassical analysis. See Anantharaman-Léautaud '14 for details.

Semiclassical limits of quasimodes: I

- Reduction to quasimode: If (1) is not true, then there exists a sequence of quasimodes (ψ_h) such that

$$P_{h,a}\psi_h := (-h^2\Delta - 1 + iha(x))\psi_h = o_{L^2}(h^{\alpha+1}), \quad \|\psi_h\|_{L^2} = 1.$$

- Semi-classical limits of (ψ_h) are Radon measures on the phase space T^*M.

For the moment, consider a sequence of $o(h)$ quasi-modes:

$$P_{h,a}\psi_h := (-h^2\Delta - 1 + iha(x))\psi_h = o_{L^2}(h).$$

• Existence of semiclassical measures (Gérard, Tartar): By extracting a subsequence of $(\psi_h)_{h>0}$, there exists a Radon probability measure μ, such that for any symbol $b \in C_c^\infty(T^*M)$,

$$\langle \mathrm{Op}_h(b)\psi_h, \psi_h\rangle_{L^2(M)} \to \int_{T^*M} b(x,\xi)\mu(dxd\xi),$$

where $\mathrm{Op}_h(b)$ is the standard quantization of the symbol b.

Semiclassical limits: II

The fact that ψ_h are $o(h)$ quasimodes implies:

- $\mathrm{supp}(\mu) \subset S^*M := T^*M \cap \{|\xi|_g = 1\}$.
- μ is invariant along the geodesic flow φ_t on S^*M.
- $a\mu \equiv 0$. This can be deduced from the a priori estimate:

$$\langle a(x)\psi_h, \psi_h \rangle \leq h^{-1}\mathrm{Im}|\langle \psi_h, P_{h,a}\psi_h \rangle|.$$

In summary, the resolvent estimate we want to study is equivalent to prove the following statement:

- (S): There is no semiclassical measure μ for $o(h^{1+\alpha})$ quasimodes ψ_h of $P_{h,a}$.

And the optimality of the resolvent esitimate is equivalent to find some $O(h^{1+\alpha})$ quasimodes.

• At this stage, it turns out that the global geometry of M and the vanishing order of $a(x)$ play definitive roles.

Geometric Control Condition (GCC)

Theorem (Rauch-Taylor ($\partial M = \emptyset$), Bardos-Lebeau-Rauch ($\partial M \neq \emptyset$))

Assume that $a \in C(M)$ *and* $\omega := \{x : a(x) > 0\}$ *satisfies the* **geometric control condition** *(GCC), then we have (S) with* $\alpha = 0$*. Equivalently, we have the resolvent estimate*

$$\|P_{h,a}^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-1}.$$

By Gearhart's theorem, the energy of the damped wave equation decays exponentially: $E[u](t) \leq Ce^{-ct}E[u](0)$ *for all* $t \geq 0$*.*

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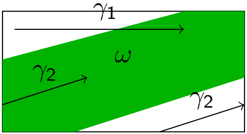
By Gearhart's theorem, the energy of the damped wave equation decays exponentially: $E[u](t) \leq Ce^{-ct}E[u](0)$ for all $t \geq 0$.

ω satisfies (GCC) means: there exists $T_0 > 0$, such that every geodesics of length $T > T_0$ passes through ω.

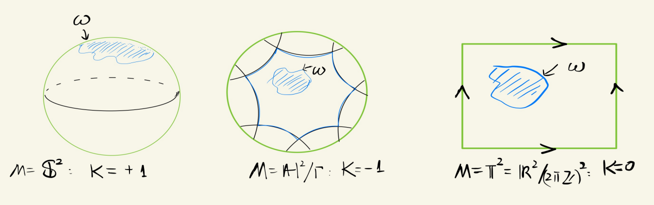


Example of $\mathbb{T}^2$: GCC is satisfied

Beyond GCC: Three model surfaces (compact)

- When $\overline{\omega}$ violates (GCC), the resolvent bound cannot be $O(h^{-1})$, and we are obliged to analyze finer quasi-modes $o(h\epsilon_h)$, $\epsilon_h \to 0$.
- Quasi-modes with smaller width (in order) $o(h\epsilon_h)$ requires finer microlocalization along the trapped geodesics. At this stage, the dynamical property of the geodesic flow φ_t plays an important role!

Beyond GCC: Three model surfaces (compact)



ω violates (GCC), $\|P_{h,b}^{-1}\|_{L^2(M)} \sim g(h)$		
	energy decay rate $f(t)$	resolvent bound $g(h)$
$M = \mathbb{S}^2$	$1/\log(1+t)$	$e^{c/h}$ [1]
$M = \mathbb{H}^2/\Gamma$	$e^{-c't}$ losing derivatives	$h^{-1}\log(h^{-1})$ [2]
$M = \mathbb{T}^2$	$t^{-1/\alpha}$	$h^{-1-\alpha}$, α close to 1 for regular a [3][4][5][6]

References:

[1] Lebeau (1996), [2] Dyatlov-Jin (2018), [3] Burq-Hitrik (2005), [4] Anantharaman-Léautaud (2014), [5] Datchev-Kleinhenz (2020), [6] S. (2021).

Damped wave operator on $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$

Denote by

$$f(t) = \sup_{(u_0,v_0)\in H^2\times H^1} \frac{\|e^{t\mathcal{A}}(u_0,v_0)\|_{\mathcal{H}}}{\|(u_0,v_0)\|_{H^2\times H^1}}.$$

the optimal decay rate of semi-group.

- When (GCC) is violated by $\overline{\omega}$, where $\omega = \{a > 0\}$ and $\overline{\omega} \neq \mathbb{T}^2$, one cannot expect that the order of $f(t)$ is better than $O(t^{-1})$.
- It turns out that the optimal order of $f(t)$ depends on the vanishing behavior of $a(x)$ near $\Sigma := \partial\{a > 0\}$:
  - Anantharaman-Léautaud 2014: For general $a \in W^{N,\infty}$ and $|\nabla a| \lesssim a^{1-\frac{1}{\beta}}$ with $N, \beta \gg 1$, $f(t) \lesssim t^{-1+\frac{C}{\beta}}$.
    (Earlier contribution by Burq-Hitrik for 1D damping)

  - Datchev-Kleinhenz 2020: When $a(x) = a(x_1) \sim x_1^{\beta}$ near $x_1 = 0$ and $\overline{\omega} \neq \mathbb{T}^2$, $f(t) \sim t^{-1+\frac{1}{\beta+3}}$, optimal (for all $\beta \geq 0$)!
    (Earlier contribution by R. Stath for the case $\beta = 0$)
- S.: It turns out the curvature of $\Sigma = \partial\{a > 0\}$ also plays a role for the optimal decay rate. In particular, the strictly convex damping region can better stabilize the wave!

The Strictly Convex damping region

Theorem (C. S., IMRN, 2022)

Let $\beta > 4$. Assume that the damping $a \geq 0$ has nice vanishing property and the open set $\omega := \{z \in \mathbb{T}^2 : a(z) > 0\}$ is locally strictly convex. Assume that $a(z)$ is locally Hölder of order β near $\partial\omega$, in the sense that there exists $R_0 > 1$, such that

$$\frac{1}{R_0}\mathrm{dist}(z, \partial\omega)^{\beta} \leq a(z) \leq R_0 \mathrm{dist}(z, \partial\omega)^{\beta}, \quad \text{for } z \in \omega \text{ near } \partial\omega.$$

Then the energy decay rate of the damped wave equation satisfies $f(t) \leq Ct^{-1+\frac{2}{2\beta+7}}$. Moreover, there are examples of strictly convex dampings $a(x)$ such that the above rate is saturated.

- The decay rate is better than the optimal decay rate $t^{-1+\frac{1}{\beta+3}}$ for the rectangular-shaped ω.

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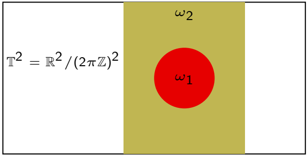

The damping a_1 generates better decay rate than a_2

$a_1(x) = (0.1 - |(x_1, x_2)|)_+^{\beta}$, $a_2(x) = (0.5 - |x_1|)_+^{\beta}$

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- The reason behind is the averaging property of $a(x)$ along the geodesic flow. As the support of a is strictly convex, we gain $1/2$ Hölder regularity locally near the place where $a = 0$.
- The proof is different from the proof in the article of Anantharaman-Léautaud.

Reduce to a second microlocalization problem

- Assume that the quasi-modes u_h satisfy

$$(-h^2\Delta - 1 + iha)u_h = o_{L^2}(h^{2+\delta}).$$

- A priori estimates:

$$\int_{\mathbb{T}^2} |h\nabla u_h|^2 - |u_h|^2 = o(h^{2+\delta}), \quad \int_{\mathbb{T}^2} a|u_h|^2 = o(h^{1+\delta}).$$

- Semiclassical measures μ of (u_h) are invariant along the geodesic flow

on $\mathbb{T}^2_{x,y} \times \mathbb{S}^2_{\xi,\eta}$. Hence $\mu = 0$ when restricting on irrational directions. To obtain a contradiction, it suffices to show that $\mu|_{\mathbb{T}^2 \times \Xi} = 0$ for any rational direction Ξ.

- W.L.O.G, we assume that trapped direction is $(0,1)$ and the quasi-modes are localized at scales $|D_y| \sim h^{-1}, |D_x| \ll h^{-1}$.

- We have to do a second microlocalization according to the size of $|D_x|$:
  - Transversal high frequencies (TH): $|D_x| \gtrsim h^{-\frac{1+\delta}{2}}$
  - Transversal low frequencies (TL): $|D_x| << h^{-\frac{1+\delta}{2}}$

High-level summary of the proof

• (TH) contributions can be ruled out by a positive commutator method.

• Using the Birkhoff norm form to average the damping and reduce it to dimension 1 in the TL regime (inspired by the work of Hitrik-Sjöstrandand and Burq-Zworski).

• Using the known sharp resolvent estimate for the 1D damping (by Datchev-Kleinhenz).

• Key geometric point: averaging along any direction for the convex-shaped damping will gain $1/2$ vanishing order.

• The construction of quasi-modes that saturated the optimal resolvent bound follows from the inverse of the above procedure.

The averaging method

•Ansatz: Find $Q_h := q(x, y, hD_y)$ and conjugate with the principal operator

$$P_{h,a} := -h^2\Delta - 1 + iha(x, y)$$

and change the quasimodes: $u_h \mapsto e^{Q_h}u_h$.

$$\begin{aligned} e^{Q_h}P_{h,a}e^{-Q_h} =& P_{h,a} - ih \cdot \frac{i}{h}[Q_h, P_{h,a}] + \frac{1}{2}[Q_h, [Q_h, P_{h,a}]] + O_{\mathcal{L}(L^2)}(h^3) \\ =& -h^2\Delta - 1 + iha + ih\mathrm{Op}_h(2\eta\partial_y q) + O_{\mathcal{L}(L^2)}(h^3) \\ &+ih\mathrm{Op}_h(2\xi\partial_x q) + ih[Q_h, a] + \frac{1}{2}[Q_h, [Q_h, P_{h,a}]] \end{aligned}$$

• Solving the cohomological equation (up to a cutoff away from $\eta = 0$)

$$2\eta\partial_y q = -(a - \mathcal{A}(a)(x)), \quad \mathcal{A}(a)(x) := \int_{\mathbb{T}} a(x, y)dy.$$

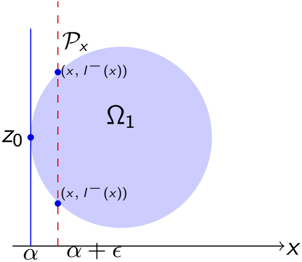
• The red terms are priorly NOT remainders! What saves us is the a priori estimate

$$\|a^{\frac{1}{2}}(e^{Q_h}u_h)\|_{L^2} = o(h^{\frac{1+\delta}{2}})$$

and a bootstrap argument yields also

$$\left\|\mathcal{A}(a)^{\frac{1}{2}}(e^{Q_h}u_h)\right\|_{L^2} = o(h^{\frac{1+\delta}{2}}).$$

Gain of vanishing order



Averaging improves the vanishing order

More precisely, if $\mathcal{A}(a)$ vanishes near $x_0 = \alpha$ and suppose that $\alpha \vec{z}_0$ is tangent to Σ_a, where $z_0 = (\alpha, y_0)$, then

$$x - \alpha \sim c(y - y_0)^2.$$

Then for $x > \alpha$ near α, each vertical line $\mathcal{P}_x$ intersects Σ_a at two points $(x, l^-(x)), (x, l^+(x))$, with

$$|l^+(x) - l^-(x)| \sim \sqrt{x - \alpha}.$$

As $a(z)$ has vanishing order β, the averaged function $\mathcal{A}(a)$ is comparable to

$$\mathcal{A}(a)(x) \sim |l^+(x) - l^-(x)| \cdot |x - \alpha|^\beta \sim |x - \alpha|^{\beta + \frac{1}{2}}.$$

Perspectives

- We still do not know a simple criterion of the optimal decay rate for the damped wave equation on $\mathbb{T}^2$ in terms of the damping $a(x)$.
- Rough damping case? For example, $a(x) = \mathbf{1}_{\Omega}(x)$, where Ω is strictly convex.
- High dimensional torus ($d \geq 3$) without GCC: The Schrödinger observability result of Anantharaman-Macià provides a decay rate $t^{-\frac{1}{2}}$, but this should not be optimal!

Thank you for your attention!