§5. Termes principaux

$$\mathcal{O}_t^J := \mathbb{E} \int d\xi\, dv\, \overline{\widehat{J}_\varepsilon(\xi,v)}\, \widehat{W}_{\psi_M^{main}(t)}(\xi,v)$$

$$= \sum_{n,n'=0}^{M-1} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi \text{ simple}}} \left[C_\pi^{\widehat{J}_\varepsilon} + \mathcal{O}\left(\bar{n}! \frac{(C\lambda^2 t)^{\bar{n}}}{t^\theta} (\log t)^{\bar{n}+3}\right)\right]$$

$\uparrow$ Lemme 4.2 – Lemme 4.3

$$t = \mathcal{O}\left(\frac{1}{\varepsilon}\right), \quad \bar{n} = \frac{n+n'}{2} \leq n_0(\varepsilon) = \frac{\gamma |\ln \varepsilon|}{\ln|\ln \varepsilon|}$$

$$\lim_{\varepsilon \to 0^+} \mathcal{O}_t^J = \lim_{\varepsilon \to 0^+} \sum_{n,n'=0}^{M-1} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \text{simple}}} C_\pi^{\widehat{J}_\varepsilon}$$

- On a besoin des notations supplémentaires adaptées aux graphes simples.

$\pi \in \Pi_{n,n'}$ simple peut-être caractérisé par: $m \in \mathbb{N}^*$, $\vec{\gamma}_m = (\gamma_0, \gamma_1, \cdots, \gamma_m) \in (\mathbb{R}^d)^{m+1}$

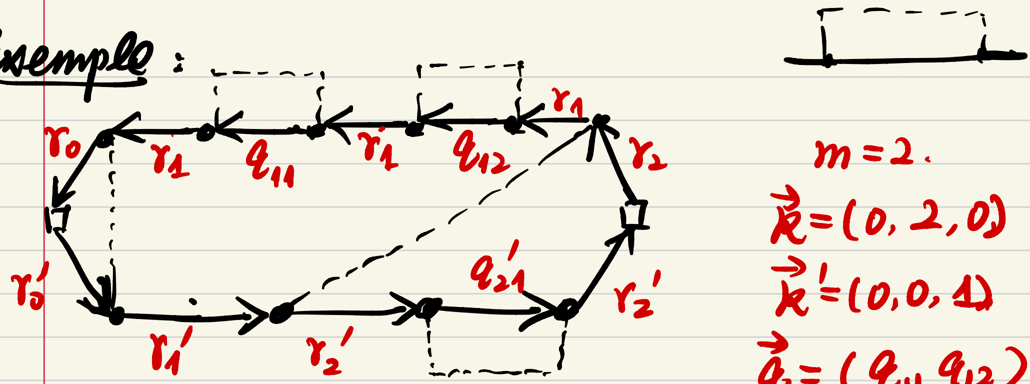
$$\vec{\gamma}_m' = (\gamma_0', \gamma_1', \cdots, \gamma_m').$$

$$\vec{k} = (k_0, k_1, \cdots, k_m) \in \mathbb{N}^{m+1}; \quad \vec{k}' = (k_0', \cdots, k_m').$$

et $\quad \vec{q} = (q_{j\ell})_{0 \leq j \leq m,\ 1 \leq \ell \leq k_j}$, si $k_j \geq 1$

ou $\underline{\vec{q}} = \phi$. si $k_j = 0$.

Exemple :



$m = 2$.
$\vec{k} = (0, 2, 0)$
$\vec{k}' = (0, 0, 1)$
$\underline{\vec{q}} = (q_{11}, q_{12})$
$\underline{\vec{q}}' = (q_{21}')$
$\vec{r} = (r_0, r_1, r_2)$
$\vec{r}' = (r_0', r_1', r_2')$

En général, on note

$$|\vec{k}| = m = \dim + 1$$

$$\|\vec{k}\| = \sum_{j=0}^{m} k_j$$

$$n = m + 2\|\vec{k}\|, \qquad n' = m + 2\|\vec{k}'\|$$

(comptage simple)

$$C_\pi^{\hat{J}_\varepsilon} = \lambda^{2\bar{n}} \int d\xi \, dv_0 \, \hat{J}_\varepsilon(\xi, v_0) \int d\vec{P}_n \, d\vec{P}_{n'}' \, \overline{K(t, \vec{p}, n)} K(t, \vec{p}', n')$$
$$\cdot \Delta_\pi(\vec{P}, \vec{P}') \, \delta(p_0 - v_0 + \tfrac{\xi}{2}) \, \delta(p_0' - v_0 - \tfrac{\xi}{2})$$
$$\cdot F(\vec{p}, n) \, F(\vec{p}', n')$$

$$d\vec{P}_n \, d\vec{P}_{n'}' = d\vec{r}_m \, d\underline{\vec{q}} \cdots$$

$$r_j = v_j - \frac{\xi}{2}. \qquad r_j' = v_j + \frac{\xi}{2}. \qquad v = v_0$$

$$\vec{r}_m = \vec{v}_m - \frac{\vec{\xi}}{2}. \qquad \vec{r}_m' = \vec{v}_m + \frac{\vec{\xi}}{2}.$$

$$C_\pi^{\hat J_\varepsilon} = \lambda^{2\bar n} \int d\xi \, d\vec v_m \hat J_\varepsilon(\xi, v_0) \int d\underline{\vec q} \cdot \overline{Q}(t, \vec v_m - \tfrac{\vec \xi}{2}, \underline{q}, \vec k) \cdot Q(t, \vec v_m + \tfrac{\vec \xi}{2}, \underline{q}', \vec k')$$

$$(*) \qquad \times \overline{\underline{P}}(\vec v_m - \tfrac{\vec \xi}{2}, \underline{q}, \vec k) \, \underline{P}(\vec v_m + \tfrac{\vec \xi}{2}, \underline{q}', \vec k')$$

$$\times \overline{M}(\vec v_m - \tfrac{\vec \xi}{2}) \, M(\vec v_m + \tfrac{\vec \xi}{2}) \, \widehat{W}_{\psi_0}(\xi, v_m)$$

$$\left(= \overline{\hat\psi_0}(v_m - \tfrac{\xi}{2}) \hat\psi_0(v_m + \tfrac{\xi}{2}) \right.$$

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où $Q(t, \vec r, \underline{q}, \vec k) = K(t, \vec p, n)$, ...

$$\underline{P}(\vec r, \vec{\underline{q}}, \vec k) = \prod_{j=0}^{m} \prod_{\ell=1}^{k_j} R(r_j - q_{j\ell})$$

$$M(\vec r_m) = \prod_{j=0}^{m-1} R(r_j - r_{j+1})$$

$$Q(t, \vec r, \underline{q}, \vec k) = K(t, \underbrace{r_0, \dots, r_0}_{k_0+1}, \underbrace{q_{01}, \dots, q_{0k_0}}_{k_0}, \dots, \underbrace{r_m, \dots, r_m}_{k_m+1}, q_{m1}, \dots, q_{mk_m})$$

$$= (-i)^m \underbrace{\int_0^t \cdots \int_0^t}_{m+1} \delta\left(t - \sum_{j=0}^m t_j\right) dt_0 \cdots dt_m \times \prod_{j=0}^m K(t_j, \underbrace{r_j, \dots, r_j}_{k_j+1}, q_{j1}, \dots, q_{jk_j})$$

$$= e^{t\eta} \int_{-\infty}^{\infty} d\alpha \, e^{-i\alpha t} \prod_{j=0}^m \left(\frac{1}{\alpha - \frac{r_j^2}{2} + i\eta} \right)^{k_j+1} \prod_{j=0}^m \prod_{\ell=1}^{k_j} \left(\frac{1}{\alpha - \frac{q_{j\ell}^2}{2} + i\eta} \right)$$

plugin... on note

$$\int d q_{je} \, \frac{R(r_j - q_{je})}{\alpha - \frac{q_{je}^2}{2} + i\eta} =: H_{\alpha,\eta}(r_j)$$

$$\Rightarrow C_\pi^{\hat{J}_\varepsilon} = \lambda^{2\bar{n}} \int d\xi \, d\vec{v}_m \, \overline{\hat{J}_\varepsilon(\xi, v_0)} \iint d\alpha \, d\beta \, e^{i(\alpha-\beta)t}.$$

$$\times \prod_{j=0}^{m} \left[\overline{H_{\alpha,\eta}\left(v_j - \frac{\xi}{2}\right)}\right]^{k_j} \cdot \left[H_{\beta,\eta}\left(v_j + \frac{\xi}{2}\right)\right]^{k_j'}.$$

$$\times \prod_{j=0}^{m} \left(\frac{1}{\alpha - \frac{(v_j - \frac{\xi}{2})^2}{2} - i\eta}\right)^{k_j+1} \left(\frac{1}{\beta - \frac{(v_j + \frac{\xi}{2})^2}{2} + i\eta}\right)^{k_j'+1}$$

$$\times \overline{M}\left(\vec{v}_m - \frac{\vec{\xi}}{2}\right) M\left(\vec{v}_m + \frac{\vec{\xi}}{2}\right) \cdot W_{\psi_0}(v_m, \xi)$$

$d \geq 3$:

$$\| H_{\alpha,\eta}(r) \|_{L^\infty_r} \leq C.$$

$$| H_{\alpha,\eta}(r) - H_{\alpha,\eta}(r') | \leq C |r - r'|.$$

$$H_{\alpha,0}(r) := \lim_{\eta \to 0^+} H_{\alpha,\eta}(r) \text{ existe.}$$

Asymptotique : $|H_{\alpha,\eta}(r_j) - H_{\frac{r_0^2}{2},0}(r_0)| \leq C(|\alpha| + \eta)^{-\frac{1}{2}\,-\frac{1}{2}}(|\alpha - \frac{r_j^2}{2}| + |\alpha - \frac{r_0^2}{2}|) + C\eta^{1/2}.$

Done.

$$\prod_{j=0}^{m}\left(\overline{\Theta}_{\alpha,\eta}(r_j)\right)^{k_j}\left(\Theta_{\beta,\eta}(r_j')\right)^{k_j'}\overline{M}(\vec{r}_m)M(\vec{r}_m')$$

$$=\prod_{j=0}^{m}\left(\overline{\Theta}_{\frac{v_0^2}{2},0}(v_0)\right)^{k_j}\left(\Theta_{\frac{v_0^2}{2},0}(v_0)\right)^{k_j'}|M(\vec{v}_m)|^2$$

$$\mathcal{O}\left(\eta^{-\frac{4}{5}}+|\alpha|^{-1/2}+|\beta|^{-1/2}\right)\mathcal{O}\left(|\alpha-\frac{v_j^2}{2}|+|\beta-\frac{v_j^2}{2}|+|\xi|+|\eta|\right)$$

$$C_\pi^{\hat{\mathcal{J}}_\varepsilon}=C_\pi^{\hat{J}_{\varepsilon,main}}+o(\Delta),\ \text{where}$$

$$C_\pi^{\hat{J}_{\varepsilon,main}}=\lambda^{2n}\int d\xi\, d\vec{v}_m\ \overline{\hat{J}_\varepsilon(\xi,v_0)}\cdot\hat{W}_{\psi_0}^{\varepsilon}(\xi,v_m)\prod_{j=0}^{m}\left(\overline{\Theta}_{\frac{v_0^2}{2},0}(v_0)^{k_j}\cdot\Theta_{\frac{v_0^2}{2},0}(v_0)^{k_j'}\right)$$
$$\cdot|M(\vec{v}_m)|^2\ \overline{Q^*}(t,\vec{v}_m-\frac{\vec{\xi}}{2},\vec{k})\,Q^*(t,\vec{v}_m+\frac{\vec{\xi}}{2},\vec{k}')$$

$$=\lambda^{2(m+\|\vec{k}\|+\|\vec{k}'\|)}\int d\xi\, d\vec{v}_m\ \overline{\hat{J}_\varepsilon(\xi,v_0)}\ \overline{\Theta}_{\frac{v_0^2}{2},0}^{\|\vec{k}\|}(v_0)\cdot\Theta_{\frac{v_0^2}{2},0}^{\|\vec{k}'\|}(v_0)$$
$$\hat{W}_{\psi_0}^{\varepsilon}(\xi,v_m)|M(\vec{v}_m)|^2\cdot\overline{Q^*}(t,\vec{v}_m-\frac{\vec{\xi}}{2},\vec{k})\,Q^*(t,\vec{v}_m+\frac{\vec{\xi}}{2},\vec{k}')$$

où $Q^*(t,\vec{r}_m,\vec{k}) = K(t, \underbrace{r_0, \cdots, r_0}_{k_0+1}, \cdots, \underbrace{r_m, \cdots, r_m}_{k_m+1})$

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0^+} \sum_{n,n'=0}^{M-1} \sum_{\substack{\pi\in\Pi_{n,n'} \\ \text{simple}}} C_\pi^{\hat{J}_\varepsilon,\text{main}}$$

$$= \lim_{\varepsilon\to 0^+} \sum_{m=0}^{\infty} \sum_{\substack{\pi \text{ simple} \\ |\pi|=|\vec{k}|=|\vec{k}'|=m}} C_\pi^{\hat{J}_\varepsilon,\text{main}}$$

$$\sum_{|\vec{k}|=|\vec{k}'|=m} \lambda^{2\|\vec{k}\|} \, \text{Ⓗ}_{\frac{V_0^2}{2},0}(V_0)^{\|\vec{k}\|} \cdot Q^*(t,\vec{r}_m,\vec{k})$$

$$= \sum_{k_0,k_1,\cdots,k_m} \left(\lambda^2 \text{Ⓗ}(V_0)\right)^{k_0+\cdots+k_m} \underbrace{\int_0^t \cdots \int_0^t}_{m+1} dt_0 \cdots dt_m \, \delta(t-\sum_{j=0}^m t_j) \prod_{j=0}^m \underbrace{K(t_j, r_j, \cdots, r_j)}_{= \frac{(-it_j)^{k_j} \cdot e^{-\frac{i r_j^2 t_j}{2}}}{k_j!}}$$

$$= \underbrace{\int_0^t \cdots \int_0^t}_{m+1} \delta(t-\sum_j t_j)\, dt_0 \cdots dt_m \sum_{k_0,\cdots,k_m} \left[\prod_{j=0}^m (-i\lambda^2 t_j \text{Ⓗ}(V_0))^{k_j} \cdot \frac{e^{-i r_j^2 t_j/2}}{k_j!} \right]$$

$$= \int_0^t \cdots \int_0^t (\cdots) \prod_{j=0}^{m} \bar{e}^{\frac{i t_j r_j^2}{2} - i t_j \lambda^2 \Theta(v_0)}$$

$$= \bar{e}^{i\lambda^2 t \Theta(v_0)} K(t, \vec{r}_m)$$

$$\Rightarrow$$

$$\sum_{m} \sum_{|\vec{k}|=|\vec{\tilde{k}}|=m} C_{\pi}^{\hat{J}_{\varepsilon}, main} = \sum_{m \geq 0} \lambda^{2m} \int d\xi \, d\vec{v}_m \, \bar{\hat{J}}_{\varepsilon}(\xi, v_0)$$

$$\times e^{i\lambda^2 t (\bar{\Theta}(v_0) - \Theta(v_0))} |M(\vec{v}_m)|^2$$

$$\times \bar{K}(t, \vec{v}_m - \tfrac{\vec{\xi}}{2}) K(t, \vec{v}_m + \tfrac{\vec{\xi}}{2}) \hat{W}_{\psi_0}(\xi, v_m)$$

On peut simplifier

$$\bar{K}(t, \vec{v}_m - \tfrac{\vec{\xi}}{2}) K(t, \vec{v}_m + \tfrac{\vec{\xi}}{2})$$

$$= 2^m \underbrace{\int_0^t \cdots \int_0^t}_{m+1} \delta(t - \sum_{j=0}^{m} a_j) \int_{-a_0}^{a_0} db_0 \cdots \int_{-a_{m-1}}^{a_{m-1}} db_{m-1} \, \mathbb{1}_{|\sum_{j=0}^{m-1} b_j| \leq a_m}$$

$$\times \exp\left[-i\xi \cdot \sum_{j=0}^{m} a_j v_j + i \sum_{j=0}^{m-1} b_j \cdot (v_j^2 - v_m^2) \right]$$

plugin. $(\lambda^2 t = T)$

$$\rightsquigarrow \sum_{m\geq 0} \lambda^{2m} \cdot 2^m \int d\xi \int d\vec{v}_m \, e^{2T \operatorname{Im} \Theta(v_0)} \, \overline{\hat{J}_\varepsilon}(\xi, v_0) \widehat{W}^\varepsilon_{\psi_0}(\xi, v_m)$$

$$\times |M(\vec{v}_m)|^2 \underbrace{\int_0^t \cdots \int_0^t}_{m+1} \delta\Big(t - \sum_{j=0}^m a_j\Big) \int_{-a_0}^{a_0} \cdots \int_{-a_{m-1}}^{a_{m-1}} db_0 \cdots db_{m-1} \cdot \mathbb{1}_{\left|\sum_{j=0}^{m-1} b_j\right| \leq a_m}$$

$$\cdot \exp[\cdots\cdots]$$

$$\cdot \int \overline{\hat{J}_\varepsilon}(\xi, v_0) \widehat{W}_{\psi_0}(\xi, v_m) e^{-i\xi \cdot \sum_{j=0}^m a_j v_j} d\xi$$

$$= \int \overline{J}(\varepsilon x, v_0) W_{\psi_0}\Big(x - \sum_{j=0}^m a_j v_j, v_m\Big) dx$$

$$= \int \overline{J}(X, v_0) W_{\psi_0}\Big(\frac{X}{\varepsilon} - \sum_{j=0}^m a_j v_j, v_m\Big) \frac{dX}{\varepsilon^d}$$

$$= \int \overline{J}(X, v_0) W^\varepsilon_{\psi_0}\Big(X - \sum_{j=0}^m \varepsilon a_j v_j, v_m\Big) dX$$

$$t \to \frac{T}{\varepsilon}. \qquad \alpha_j = \varepsilon a_j.$$

on a

$$\sum_{m\geq 0} \sum_{|\vec{k}|=|\vec{k}'|=m} C_\pi^{\hat{J}_{\varepsilon,\text{main}}}$$

$$= \sum_{m\geq 0} \int dX \int d\vec{v}_m \, e^{2T\,\mathrm{Im}\,\Theta(v_0)} |M(\vec{v}_m)|^2 \cdot 2^m$$

$$\times \underbrace{\int_0^T \cdots \int_0^T}_{m+1} d\alpha_0 \cdots d\alpha_m \, \delta\Big(T - \sum_{j=0}^m \alpha_j\Big)$$

$$\times \prod_{j=0}^{m-1} \int_{-\frac{\alpha_j}{\varepsilon}}^{\frac{\alpha_j}{\varepsilon}} db_j \cdot e^{i\sum_{j=0}^m b_j (v_j^2 - v_m^2)} \mathbb{1}_{\left|\sum_{j=0}^{m-1} b_j\right| \leq \frac{\alpha_m}{\varepsilon}}$$

$$\times \bar{J}(X, v_0) \cdot \underbrace{W^\varepsilon_{\psi_0}\Big(X - \sum_{j=0}^m \alpha_j v_j, v_m\Big)}_{\to F_0\left(X - \sum_{j=0}^m \alpha_j v_j, v_m\right)}$$

$$\xrightarrow{\varepsilon \to 0^+} \prod_{j=0}^{m-1} 2\pi\, \delta(v_j^2 - v_m^2)$$

$\Rightarrow \quad \varepsilon \to 0^+$

$$\sum_{m\geq 0} \sum_{|\vec{k}|=|\vec{k}'|=m} C_\pi^{\hat{J}_{\varepsilon,\text{main}}}$$

$$\xrightarrow[\varepsilon\to 0^+]{} \sum_{m\geq 0} \int dX\, dv_0\, e^{2T\,\mathrm{Im}\,\Theta(v_0)} \cdot \bar{J}(X,v_0)$$

$$\int dv_1 \cdots dv_m \Big(\prod_{j=0}^{m-1} \underbrace{4\pi R(v_j - v_{j+1})\,\delta(v_j^2 - v_{j+1}^2)}_{\sigma(v_j,\, v_{j+1})} \Big)$$

$$\times \underbrace{\int_0^T \cdots \int_0^T}_{m+1} d\alpha_0 \cdots d\alpha_m\, \delta\Big(T - \sum_{j=0}^{m} \alpha_j\Big) F_0\Big(X - \sum_{j=0}^{m} \alpha_j v_j, v_m\Big)$$

$$2T\,\mathrm{Im}\,\Theta(v_0) = -T \int \sigma(U,V)\,dU$$
$$= -T\,\sigma_d(V)$$

$\Rightarrow$

$$\lim_{\varepsilon\to 0^+} W^{\varepsilon}_{\psi_{\frac{T}{\varepsilon}}}(X,V_0)$$

$$= \sum_{m\geq 0} e^{-T\sigma_0(V_0)} \int dV_1 \cdots dV_m\, \sigma(V_0,V_1)\cdots\sigma(V_{m-1},V_m)$$

$$\times \underbrace{\int_0^T \cdots \int_0^T}_{m+1} \delta\Big(T-\sum_{j=0}^m \alpha_j\Big)\, d\alpha_0 \cdots d\alpha_m\, F_0\Big(X-\sum_{j=0}^m \alpha_j V_j, V_m\Big)$$

La série Dyson de Boltzmann.

Boltzmann : $\Phi_s^V(X) = X - sV$:

$$F_T(X,V) = e^{-T\sigma_0(V)} F_0\big(\Phi_T^V(X), V\big)$$

$$+ e^{-T\sigma_0(V)} \int_0^T d\tau \int \sigma(U,V)\, e^{\tau\sigma_0(V)} F_\tau\big(\Phi_{T-\tau}^V(X), U\big)\, dU$$

$\Rightarrow$ On termine la dérivation de Boltzmann linéaire. $\square$