GIBBS MEASURE FOR THE FOCUSING QUINTIC NLS

In this note, we review the work of *Lebowitz-Rose-Speer* for the construction of the Gibbs measure ([3]) for the focusing NLS on the circle:

ocusingNLS

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^4 u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u|_{t=0} = u_0 \end{cases} \tag{0.1}$$

via the variational method of *Boué-Dupuis* and popularized by *Barashkov-Gubinelli.* The note is self-contained except for the variational formula Proposition 3.1.

1. Gaussian free field

The Gaussian measure γ_1 is the law of the random Fourier series

$$\phi^\omega(x) := \sum_{k \in \mathbb{Z}} \frac{g_k(\omega) \mathrm{e}^{ikx}}{\sqrt{1+k^2}},$$

where $(g_k(\omega))_{k \in \mathbb{Z}}$ are i.i.d. standard complex gaussians, that is

$$g_k(\omega) = \frac{X_k(\omega) + iY_k(\omega)}{\sqrt{2}}, \quad X_k, Y_k \sim \mathcal{N}_{\mathbb{R}}(0,1)$$

and X_k, Y_k are mutually independent. Note that γ_1 has the covariance operator $(1-\Delta)^{-1}$, and formally, $d\gamma_1(u) = \frac{1}{\mathcal{Z}_1} \mathrm{e}^{-\frac{1}{2}\|u\|_{H^1}^2} du$.

For fixed $N \in \mathbb{Z}$, set

$$\phi_N^\omega(x) := \Pi_N \phi^\omega := \sum_{|k| \le N} \frac{g_k(\omega) \mathrm{e}^{ikx}}{\sqrt{1+k^2}},$$

where Π_N is the Fourier projector such that $\Pi_N \Big(\sum_{k \in \mathbb{Z}} f_k \mathrm{e}^{ikx} \Big) = \sum_{|k| \le N} f_k \mathrm{e}^{ikx}$.

Fix $\sigma \in (0, \frac{1}{2})$. We have

$$\mathbb{E}[\|\phi_N - \phi\|_{H^\sigma}^2] \le C N^{-(1-2\sigma)} \to 0,$$

by the Chebyshev's inequality, for any $\epsilon_0 > 0$,

$$\mathbb{P}\{\|\phi_N^\omega - \phi^\omega\|_{H^s} > \epsilon_0\} \le C\epsilon_0^{-2} N^{-(1-2\sigma)} \to 0, \ N \to \infty.$$

Therefore, ϕ_N^ω converges in probability to ϕ^ω. In particular, the law $\gamma_{1,N}$ of ϕ_N^ω converges to γ_1.

To get an idea of the law $\gamma_{1,N}$ which can be viewed as a Gaussian measure on the finite-dimensional space $\mathbb{C}^{2N+1}$. Set $\widehat{u}_k = a_k + ib_k$ and

$$\mathrm{d}L_N(u) := \prod_{|k| \le N} \mathrm{d}\widehat{u}_k := \prod_{|k| \le N} \mathrm{d}a_k \mathrm{d}b_k$$

the Lebesgue measure on $\mathbb{C}^{2N+1}$, then viewing $(\mathrm{e}^{ikx})_{|k| \le N}$ as a basis of $\mathbb{C}^{2N+1}$, the law $\gamma_{1,N}$ can be written as

$$\mathrm{d}\gamma_{1,N}(u) := \frac{1}{\mathcal{Z}_{1,N}} \mathrm{e}^{-\sum_{|k| \le N} \frac{1}{2}(1+k)^2 |\widehat{u}_k|^2} \mathrm{d}L_N(u) = \frac{1}{\mathcal{Z}_{1,N}} \prod_{|k| \le N} \mathrm{e}^{-\frac{1}{2}(1+k^2)|\widehat{u}_k|^2} \mathrm{d}\widehat{u}_k,$$

where the normalization constant

$$\mathcal{Z}_{1,N} := \int_{\mathbb{C}^{2N+1}} \prod_{|k|\leq N} \mathrm{e}^{-\frac{1}{2}(1+k^2)\frac{a_k^2+b_k^2}{2}} \mathrm{d}a_k \mathrm{d}b_k = \prod_{|k|\leq N} \frac{4\pi}{1+k^2} = \frac{(4\pi)^{2N+1}}{(N!)^4} \prod_{|k|\leq N} \Big(1+\frac{1}{k^2}\Big)^{-1}.$$

Note that $\mathcal{Z}_{1,N} \to 0$. From the density formula of $\gamma_{1,N}$, we remark that

$$\mathrm{d}\gamma_{1,N}(u) = \frac{1}{\mathcal{Z}_{1,N}} \mathrm{e}^{-\frac{1}{2}\|\Pi_N u\|_{H^1}^2} \mathrm{d}\Pi_N u.$$

Remark 1.1. *As an exercise, one can check the Fernique theorem:*

$$\int_{H^\sigma} \mathrm{e}^{\epsilon_0 \|u\|_{H^\sigma}^2} \mathrm{d}\gamma_1 < \infty.$$

However,

$$\lim_{N\to\infty} \int_{H^\sigma} \mathrm{e}^{c_0 \|\Pi_N u\|_{H^1}^2} \mathrm{d}\gamma_{1,N} = \infty,$$

for any $c_0 > 0$.

2. The Gibbs measure and the ground state (soliton)

Note that (0.1) is a Hamiltonian system associated to the Hamiltonian

$$H[u] := \int_{\mathbb{T}} \Big(\frac{1}{2}|\partial_x u|^2 dx - \frac{1}{6}|u|^6\Big) \mathrm{d}x.$$

The Gibbs measure of (0.1) has the formal expression $\mathrm{d}\rho(u) = \frac{1}{\mathcal{Z}}\mathrm{e}^{-H[u]}\mathrm{d}u$. Due to the non-existence of the Lebesgue measure $\mathrm{d}u$, we might define $\mathrm{d}\rho(u)$ by $\mathrm{d}\rho(u) = \mathrm{e}^{\frac{1}{6}\|u\|_{L^6}^6}\mathrm{d}\gamma_1(u)$. However, it turns out that the formal density $\mathrm{e}^{\frac{1}{6}\|u\|_{L^6}^6}$ is not integrable with respect to $\mathrm{d}\gamma_1$.

Roughly speaking, to define a meaningful Gibbs measure, we require that the quadratic part (which is positive) in $H[u]$ is dominant. This is impossible without any constraint on the mass (another conserved quantity of (0.1)) $\|u\|_{L^2}$. It turns out that the threshold of the mass truncation is dictated by the sharp Gagliardo-Nirenberg inequality

$$\|f\|_{L^6(\mathbb{R})}^6 \leq C_{\mathrm{GN}}^6 \|f_x\|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^4,$$

where $C_{\mathrm{GN}} = \frac{4}{\pi^2}$. The equality is attained by the ground state $Q = Q(x)$, which is radial, positive, exponentially decreasing at infinity and is satisfied the elliptic equation

$$-Q'' + cQ - Q^5 = 0.$$

Recall that the ground state satisfies $H[Q] = 0$, hence

$$C_{\mathrm{GN}}^6 \|Q\|_{L^2(\mathbb{R})}^4 = 3.$$

So if $\|u\|_{L^2(\mathbb{R})} < \|Q\|_{L^2(\mathbb{R})}$, we have

$$\begin{aligned} H[u] =& \frac{1}{2}\|u_x\|_{L^2(\mathbb{R})}^2 - \frac{1}{6}\|u\|_{L^6(\mathbb{R})}^6 \\ \geq& \frac{1}{2}\|u_x\|_{L^2(\mathbb{R})}^2 - \frac{C_{\mathrm{GN}}}{6}\|u_x\|_{L^2(\mathbb{R})}^2\|u\|_{L^2(\mathbb{R})}^4 \\ =& \frac{1}{2}\|u_x\|_{L^2(\mathbb{R})}^2\Big(1 - \frac{\|u\|_{L^2(\mathbb{R})}^4}{\|Q\|_{L^2(\mathbb{R})}^4}\frac{\|Q\|_{L^6}^6}{3\|Q_x\|_{L^2(\mathbb{R})}^2}\Big) \\ =& \frac{1}{2}\|u_x\|_{L^2(\mathbb{R})}^2\Big(1 - \frac{\|u\|_{L^2(\mathbb{R})}^4}{\|Q\|_{L^2(\mathbb{R}^2)}^4}\Big) > 0. \end{aligned}$$

This means that when we truncate the mass by $K < \|Q\|_{L^2(\mathbb{R})}$, by the sharp Gagliardo-Nirenberg inequality, the formal measure $\nu_K(du) := \mathrm{e}^{-H[u]}\mathbf{1}_{\|u\|_{L^2}<K}du$ is dominated by the quadratic form $\mathrm{e}^{-c\|u_x\|_{L^2}^2}du$, which is normalizable. In the next section we make this heuristics rigorous.

3. Normalizable below the mass threshold of the ground state

3.1. **Variational formulation.** Let $\{B_k(\cdot)\}_{k\in\mathbb{Z}^2}$ be a collection of standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $B_k = \overline{B_{-k}}$ and otherwise independent. Let

$$X(t) = \sum_{k\in\mathbb{Z}} B_k(t)\mathrm{e}^{ikx} ,$$

which is the cylindrical Brownian motion on $L^2(\mathbb{T})$ adapted to the filtration $(\mathcal{F}_t)$ generated by $\{B_k\}$.

For every N, let $\mathcal{S}_N$ be the operator such that

eq:S_N (3.1)
$$\widehat{\mathcal{S}_N f}(k) = \frac{\widehat{f}(k)}{\langle k\rangle} \cdot \mathbf{1}_{|k|\leq N} . \tag{3.1}$$

Let $W_N(t) := \mathcal{S}_N X(t)$, and for every N, define the measure $\mathbf{Q}_N$ by

$$\frac{\mathrm{d}\mathbf{Q}_N}{\mathrm{d}\mathbf{P}} := \frac{1}{\mathcal{Z}_N}\mathrm{e}^{-\frac{1}{6}\int_{\mathbb{T}}|W_N(1)|^6\mathrm{d}x} \cdot \mathbf{1}_{[0,K]}(\|W_N(1)\|_{L^2}) .$$

Here, the integration variable in x is from $W_N(1) = W_N(1,\cdot)$. For $t = 1$, we also simply write W_N for $W_N(1)$. Then

$$\mathrm{Law}_{\mathbf{P}}\big(W_N(1)\big) = \gamma_1 ,$$

and the normalisation constant $\mathcal{Z}_N$ is the same as above.

For a space-time function v, we denote

$$\mathcal{I}_N(v) := \mathcal{S}_N\int_0^1 v(s)\mathrm{d}s.$$

In order to estimate the partition function, we will make use of the following variational formula:

ational_BD **Proposition 3.1** ([1][2]). *Let F be a real-valued bounded functional, we have*

ariational (3.2)
$$\log\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{F(W_N)}\big] = \sup_{v\in\mathbf{H}_a}\mathbb{E}^{\mathbf{P}}\Big[\mathrm{e}^{F(W_N+\mathcal{I}_N(v))} - \frac{1}{2}\int_0^1\|v(t)\|_{L^2}^2\mathrm{d}t\Big] , \tag{3.2}$$

$\mathbf{H}_a$ *denotes all predictable processes in L^2 with respect to the filtration generated by the process $X(t)$.*

3.2. **Normalizable for the case $K < \|Q\|_{L^2(\mathbb{R})}$.** Since

$$\begin{aligned}\mathcal{Z}_N =&\mathbb{E}^{\mathbf{P}}\Big[\exp\Big(\frac{1}{6}\|W_N\|_{L^6}^6\Big)\mathbf{1}_{\|W_N\|_{L^2}\leq K}\Big]\\ =&\underbrace{\mathbb{E}^{\mathbf{P}}\Big[\exp\Big(\frac{1}{6}\|W_N\|_{L^6(\mathbb{T})}^6\mathbf{1}_{\|W_N\|_{L^2}\leq K}\Big)\Big]}_{\mathcal{Z}_{N,K}}-\mathbb{P}[\|W_N\|_{L^2}>K].\end{aligned}$$

To prove the normalizable below the mass of the soliton: $K < \|Q\|_{L^2(\mathbb{R})}$, we have to show that uniformly in N,

$$+\infty > -\log \mathcal{Z}_{N,K} > -\infty.$$

Take $v \equiv 0$, we obtain easily a lower bound for $\log \mathcal{Z}_{N,K}$ by

$$-\mathbb{E}^{\mathbf{P}}\Big[\frac{1}{6}\int_{\mathbb{T}}|W_N|^6dx\Big] \leq C < \infty,$$

uniformly in N. To proceed on, we need the following version of the sharp Gagliardo-Nirenberg inequality on $\mathbb{T}$:

sharpGNT **Lemma 3.2.** *For any $\delta > 0$, there exists $C_\delta > 0$, such that for all $f \in H^1(\mathbb{T})$,*

$$\|f\|_{L^6(\mathbb{T})}^6 \leq (C_{\mathrm{GN}}+\delta)^6\|f_x\|_{L^2(\mathbb{T})}^2\|f\|_{L^2(\mathbb{T})}^4 + C_\delta\|f\|_{L^2(\mathbb{T})}^6.$$

Proof. The proof is elementary from the sharp Gagliardo-Nirenberg inequality on $\mathbb{R}$, we refer to [5] for details. □

•**Notation:** For $M \in \mathbb{N}$, denote

$$\Pi_M f := \sum_{|k|\leq M}\widehat{f}(k)\mathrm{e}^{ikx}, \quad \Pi_M^\perp f := f - \Pi_M f.$$

To obtain the upper bound of $\log \mathcal{Z}_{N,K}$, we make use of (3.2) for

$$F(W) := \frac{1}{6}\|\Pi_N W\|_{L^6}^6\mathbf{1}_{\|\Pi_N W\|_{L^2}\leq K}.$$

We remark first that by Cauchy-Schwarz,

$$\frac{1}{2}\int_0^1\|v(t)\|_{L^2}^2\mathrm{d}t \geq \frac{1}{2}\Big\|\int_0^1 v(t)\mathrm{d}t\Big\|_{L^2}^2 \geq \frac{1}{2}\|\mathcal{I}_N(v)\|_{H^1}^2.$$

Thus for any $v \in \mathbb{H}_a$,

$$\begin{aligned}&\mathbb{E}^{\mathbf{P}}\Big[\frac{1}{6}\int_{\mathbb{T}}|W_N+\mathcal{I}_N(v)|^6\mathrm{d}x\cdot\mathbf{1}_{[0,K]}(\|W_N-\mathcal{I}_N(v)\|_{L^2})+\frac{1}{2}\int_0^1\|v(t)\|_{L^2}^2dt\Big]\\ \leq&\sup_{V\in H^1}\mathbb{E}^{\mathbf{P}}\Big[\frac{1}{6}\int_{\mathbb{T}}|W_N+V|^6\mathrm{d}x\cdot\mathbf{1}_{[0,K]}(\|W_N+V\|_{L^2})-\frac{1}{2}\|V\|_{H^1}^2\Big].\end{aligned}$$

where $V = \mathcal{I}_N(v)$.

Therefore, to prove the normalizable of the Gibbs measure, we have to show that

upperbound (3.3) $$\mathbb{E}^{\mathbf{P}}\Big[\frac{1}{6}\||W_N+V\|_{L^6}^6\mathbf{1}_{[0,K]}(\|W_N+V\|_{L^2})-\frac{1}{2}\|V\|_{H^1}^2\Big] \leq C,$$

uniformly for all $V \in H^1$. Since for any $\sigma > 0$,

$$\|W_N+V\|_{L^6}^6 \leq (1+\sigma)\|V\|_{L^6}^6 + C_\sigma\|W_N\|_{L^6}^6$$

and the expectation of $\|W_N\|_{L^6}^6$ is uniformly bounded in N, it suffices to show that

oundedgoal (3.4) $$\mathbb{E}^{\mathbf{P}}[\mathrm{I}_N] := \mathbb{E}^{\mathbf{P}}\Big[\frac{1}{6}\|V\|_{L^6}^6\mathbf{1}_{[0,K]}(\|W_N+V\|_{L^2})-\frac{1}{2}\|V\|_{H^1}^2\Big]$$

is uniformly bounded in N and for $V \in H^1$. To this end, we need to estimate the first term on the right hand side. The heuristic is that, the high frequency portion for W_N is essentially very small so that we are able to apply the sharp Gagliardo-Nirenberg inequality to control it by the kinetic energy of $\Pi_M^{\perp} V$, while for the low-frequency portion, we use simply Bernstein. Now we make precise the above heuristic via a stopping time argument.

Pick $0 < \delta_0 < \|Q\|_{L^2(\mathbb{R})} - K$. Given a dyadic number $M \geq 1$, we set

$$\mathcal{E}_M := \{\|\Pi_L^{\perp} W_N\|_{L^2} > \delta_0,\ \forall L \leq M,\ \|\Pi_{2M}^{\perp} W_N\|_{L^2} \leq \delta_0\}$$

and

$$\mathcal{E}_0 := \{\|W_N\|_{L^2} \leq \delta_0\} \quad \mathcal{E}_{\frac{1}{2}} := \{\|W_N\|_{L^2} > \delta_0,\ \|\Pi_1^{\perp} W_N\|_{L^2} \leq \delta_0\}.$$

Since $\|\Pi_M^{\perp} W_N\|_{L^2} \to 0$, a.s. as $M \to \infty$, we have the decomposition

$$1 = \mathbf{1}_{\mathcal{E}_0} + \sum_{M \geq \frac{1}{2}} \mathbf{1}_{\mathcal{E}_M},\ a.s.$$

Now we estimate term by term for $M = 0, \frac{1}{2}, M \geq 1$ dyadic

INM (3.5)
$$\mathrm{I}_{N,M} := \Big(\frac{1}{6}\|V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|W_N + V\|_{L^2}) - \frac{1}{2}\|V\|_{H^1}^2\Big)\mathbf{1}_{\mathcal{E}_M} \tag{3.5}$$

For fixed $M \geq \frac{1}{2}$, we have for any small $\sigma > 0$,

splittingM (3.6)
$$\begin{aligned}
&\|\Pi_1^{\perp} V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|\Pi_1^{\perp}(W_N + V)\|_{L^2})\mathbf{1}_{\mathcal{E}_M}\\
&\quad \leq (1+\sigma)\|\Pi_{2M}^{\perp} V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|\Pi_{2M}^{\perp}(W_N + V)\|_{L^2})\mathbf{1}_{\mathcal{E}_M}\\
&\quad + C_\sigma \|\Pi_{2M} V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|\Pi_1^{\perp}\Pi_{2M}(W_N + V)\|_{L^2})\mathbf{1}_{\mathcal{E}_M}\\
&\quad \leq (1+\sigma)\|\Pi_{2M}^{\perp} V\|_{L^6}^6 \mathbf{1}_{[0,K+\|\Pi_{2M}^{\perp} W_N\|_{L^2}]}(\|\Pi_{2M}^{\perp} V\|_{L^2})\mathbf{1}_{\mathcal{E}_M}\\
&\quad + C_\sigma M^2 \|\Pi_{2M} V\|_{L^2}^6 \mathbf{1}_{[0,K+\|\Pi_{2M} W_N\|_{L^2}]}(\|\Pi_{2M} V\|_{L^2})\mathbf{1}_{\mathcal{E}_M},
\end{aligned} \tag{3.6}$$

where we used Bernstein and the fact that

$$\|\Pi_{2M}^{\perp} V + \Pi_1^{\perp}\Pi_{2M} V\|_{L^6}^6 \leq (1+\sigma)\|\Pi_{2M}^{\perp} V\|_{L^6}^6 + C_\sigma \|\Pi_{2M} V\|_{L^6}^6$$

and the fact that $\|\Pi_1^{\perp}(W_N + V)\|_{L^2} \leq K$ implies that $\|\widetilde{\Pi}\Pi_1^{\perp}(W_N + V)\|_{L^2} \leq K$ for $\widetilde{\Pi} \in \{\Pi_{2M}, \Pi_{2M}^{\perp}\}$.

For the high frequency part, we estimate using Gagliardo-Nirenberg

$$\begin{aligned}
&(1+\sigma)\|\Pi_{2M}^{\perp} V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|\Pi_{2M}^{\perp}(W_N + V)\|_{L^2})\mathbf{1}_{\mathcal{E}_M}\\
\leq&(1+\sigma)C_{\mathrm{GN}}^6 (K + \|\Pi_{2M}^{\perp} W_N\|_{L^2})^4 \|\partial_x \Pi_M^{\perp} V\|_{L^2}^2 \mathbf{1}_{\mathcal{E}_M} + C_\sigma (K + \|\Pi_{2M}^{\perp} W_N\|_{L^2})^6 \mathbf{1}_{\mathcal{E}_M}\\
\leq&(1+\sigma)C_{\mathrm{GN}}^6 (K + \delta_0)^4 \|\Pi_{2M}^{\perp} V\|_{H^1}^2 \mathbf{1}_{\mathcal{E}_M} + C_\sigma (K + \|\Pi_{2M}^{\perp} W_N\|_{L^2})^6 \mathbf{1}_{\mathcal{E}_M},
\end{aligned}$$

where to the last step, we use the fact that on $\mathcal{E}_M$, $\|\Pi_{2M}^{\perp} W_N\|_{L^2} \leq \delta_0$. Similarly, on $\mathcal{E}_0$,

$$\begin{aligned}
\|V\|_{L^6}^6 \mathbf{1}_{[0,K]}(\|W_N + V\|_{L^2})\mathbf{1}_{\mathcal{E}_0} \leq& \|V\|_{L^6}^6 \mathbf{1}_{[0,K+\delta_0]}(\|V\|_{L^2})\mathbf{1}_{\mathcal{E}_0}\\
\leq& C_{\mathrm{GN}}^6 (K + \delta_0)^4 \|V\|_{H^1}^2 \mathbf{1}_{\mathcal{E}_0} + C_\sigma (K + \delta_0)^6 \mathbf{1}_{\mathcal{E}_0}.
\end{aligned}$$

Using the fact that $\|V\|_{H^1} \geq \|\Pi_{2M}^{\perp} V\|_{H^1}$ for any dyadic $M \geq \frac{1}{2}$, we deduce that

INM' (3.7)
$$\begin{aligned}
\mathrm{I}_{N,M} \leq& \Big(\frac{(1+\sigma)^3}{6} C_{\mathrm{GN}}^6 (K + \delta_0)^4 - \frac{1}{2}\Big)\|\Pi_{2M}^{\perp} V\|_{H^1}^2 \mathbf{1}_{\mathcal{E}_M}\\
&+ C_\sigma (K + \|\Pi_{2M}^{\perp} W_N\|_{L^2})^6 \mathbf{1}_{\mathcal{E}_M} + C_\sigma M^2 (K + \|\Pi_{2M} W_N\|_{L^2})^6 \mathbf{1}_{\mathcal{E}_M},
\end{aligned} \tag{3.7}$$

where the last term comes from the Bernstein inequality

$$\|\Pi_{2M} V\|_{L^6}^6 \leq C M^2 \|\Pi_{2M} V\|_{L^2}^6 \leq C M^2 (K + \|\Pi_{2M} W_N\|_{L^2})^6$$

on $\mathcal{E}_M$. We also have

$$\mathrm{I}_{N,0} \leq \Big(\frac{1}{6}C_{\mathrm{GN}}^6(K+\delta_0)^4 - \frac{1}{2}\Big)\|V\|_{H^1}^2 \mathbf{1}_{\mathcal{E}_0} < 0$$

since $K+\delta_0 < \|Q\|_{L^2(\mathbb{R})}$. By choosing σ sufficiently small, the first term on the right hand side of (3.7) is negative. Moreover, we have

$$\sum_{M\geq \frac{1}{2}} C_\sigma K \mathbf{1}_{\mathcal{E}_M} = C_\sigma K, \ a.s.,$$

and $\mathbb{E}^{\mathbf{P}}\|W_N\|_{L^6}^6 + \mathbb{E}\|\Pi_{2M}^{\perp} W_N\|_{L^2}^6 \leq C$. So to conclude, it remains to show that

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$$\sum_{M\geq \frac{1}{2}} M^2 \mathbb{E}^{\mathbf{P}}[(K+\|\Pi_{2M}W_N\|_{L^2})^6 \mathbf{1}_{\mathcal{E}_M}] \leq C. \tag{3.8}$$

Since

$$\mathbb{E}^{\mathbf{P}}[\mathbf{1}_{\mathcal{E}_M}] \leq \mathbb{P}\{\|\Pi_M^{\perp} W_N\|_{L^2} > \delta_0\},$$

Recall the following large deviation bound:

edeviation **Lemma 3.3.** *Let X^ω is a Gaussian random field on $L^2(\mathbb{T})$, then there exist $C_0,\ c_0 > 0$ such that for any $\lambda \geq 1$,*

$$\mathbb{P}\{\|X^\omega\|_{L^2} > \lambda\} \leq C_0 \mathrm{e}^{-c_0\lambda^2 \mathbb{E}\|X\|_{L^2}^2}.$$

Proof. Write

$$X^\omega = \sum_{k\in\mathbb{Z}} c_k \cdot g_k(\omega)\mathrm{e}^{ikx},$$

we have $(c_k)_{k\in\mathbb{N}} \in l^2(\mathbb{Z})$ and $\mathbb{E}[\|X^\omega\|_{L^2}^2] = \|c_k\|_{l^2}^2$.

For any $\alpha > 0$ such that $\alpha|c_k|^2 < 1$ for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\mathbb{P}\{\|X^\omega\|_{L^2}^2 > \lambda^2\} =&\mathbb{P}\Big\{ \exp\Big(\alpha \sum_{k\in\mathbb{Z}} |c_k|^2 |g_k(\omega)|^2\Big) > \mathrm{e}^{\alpha\lambda^2}\Big\} \\ \leq& \mathrm{e}^{-\alpha\lambda^2}\mathbb{E}\Big[\exp\Big(\alpha\sum_{k\in\mathbb{Z}}|c_k|^2|g_k(\omega)|^2\Big)\Big] \\ =&\mathrm{e}^{-\alpha\lambda^2}\prod_{k\in\mathbb{Z}}\mathbb{E}\big[\mathrm{e}^{\alpha|c_k|^2|g_k(\omega)|^2}\big] \\ =&\mathrm{e}^{-\alpha\lambda^2}\prod_{k\in\mathbb{Z}}\mathrm{e}^{-\log(1-\alpha|c_k|^2)} \leq \mathrm{e}^{-\alpha(\lambda^2-2\|c_k\|_{l^2}^2)},\end{aligned}$$

where to the last inequality, we used $-\log(1-a) < 2a$ for $0 < a < 1$. □

Applying Lemma 3.3 to $X = M^{\frac{1}{2}} \cdot \Pi_M^{\perp} W_N$ with variance $O(1)$ in L^2, we deduce that

probEM
$$\mathbb{E}^{\mathbf{P}}[\mathbf{1}_{\mathcal{E}_M}] \leq \mathbf{P}[\|\Pi_M^{\perp} W_N\|_{L^2} > \delta_0] \leq C_0 \mathrm{e}^{-c_0\delta_0^2 M}. \tag{3.9}$$

By Cauchy-Schwarz, we have

$$\begin{aligned}&\sum_{M\geq 1/2} M^2 \mathbb{E}^{\mathbf{P}}[(K+\|\Pi_{2M}W_N\|_{L^2})^6\mathbf{1}_{\mathcal{E}_M}] \\ \leq& C_K \sum_{M\geq 1/2} M^2(1+(\mathbb{E}^{\mathbf{P}}[\|\Pi_{2M}W_N\|_{L^2}^{12}])^{1/2})(\mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\mathcal{E}_M}])^{\frac{1}{2}} \\ \leq& C_K \sum_{M\geq 1/2} M^2 \mathrm{e}^{-c\delta_0^2 M} \leq C.\end{aligned}$$

This proves (3.8) and finally (3.3).

4. Non-noramlizable above the mass of the ground state

In this section, we assume that $K > \|Q\|_{L^2(\mathbb{R})}$ and show that $\log \mathcal{Z}_N$ is NOT uniformly bounded.

4.1. **Rescaled soliton.** Let $\alpha \in (0,1)$ sufficiently small such that $(1+2\alpha)\|Q\|_{L^2} < K$. For $\lambda \in \mathbb{R}$, consider the rescaled soliton in $\mathbb{R}$

$$Q_{\lambda,\alpha}(x) := (1+\alpha)\lambda^{\frac{1}{2}} Q(\lambda x).$$

Direct computation yields

ledsoliton (4.1)

$$\|Q_{\lambda,\alpha}\|_{L^2(\mathbb{R})} = (1+\alpha)\|Q\|_{L^2(\mathbb{R})},\ E[Q_{\lambda,\alpha}] = -\frac{\lambda^2}{6}[(1+\alpha)^6 - (1+\alpha)^2]\|Q\|_{L^6(\mathbb{R})}^6 < 0.$$

Note that $Q_{\lambda,\alpha}$ is a candidate for proving the existence of blowup solutions for the focusing mass critical NLS on $\mathbb{R}$, via the viral argument.

Now we would like to use $Q_{\lambda,\alpha}$ as a candidate (time-independent, deterministic) of the test function in the variational formula (3.2) to build up an unbounded sequence. That is to say, we want to choose the function v in (3.2) such that $V = \mathcal{I}_N(v) = Q_{\lambda,\alpha}$. However, there are two issues for this naive choice. Firstly, $Q_{\lambda,\alpha}$ is not a function on $\mathbb{T}$. Secondly, $Q_{\lambda,\alpha}$ is not exactly frequency-localized at size smaller than N, as required from the definition $\mathcal{I}_N$. Nevertheless, when $\lambda \gg 1$, $Q_{\lambda,\alpha}$ is very concentrated near the original (of size $|x| \leq 1/\lambda$). Moreover, if $\lambda = \frac{N}{10}$, say, then the high frequency portion of $Q_{\lambda,\alpha}$ is negligible, i.e. $\Pi_N^{\perp} Q_{\lambda,\alpha} = O(N^{-A})$ for any $A > 0$. In summary, it is flexible to modify $Q_{\lambda,\alpha}$ accordingly with a small perturbation of the identities (4.1). Therefore, to avoid too much technicalities, below we still work with $Q_{\lambda,\alpha}$ and pretend that $Q_{\lambda,\alpha} = \Pi_N Q_{\lambda,\alpha}$.

We first show that the mass and energy of $Q_{\lambda,\alpha}$ are essentially concentrated at relative high frequencies. Indeed,

$$\widehat{Q}_{\lambda,\alpha}(\xi) = \frac{(1+\alpha)}{\lambda^{\frac{1}{2}}}\widehat{Q}\Big(\frac{\xi}{\lambda}\Big),$$

and we compute that

$$\|\widehat{Q}_{\lambda,\alpha}\mathbf{1}_{|\xi|\leq M}\|_{L^2(\mathbb{R})} = O\Big(\frac{M^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\Big), \quad \||\xi|\widehat{Q}_{\lambda,\alpha}\mathbf{1}_{|\xi|\leq M}\|_{L^2(\mathbb{R})} = O\Big(\frac{M^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}}\Big).$$

Therefore,

uencebound (4.2)

$$\|\Pi_M Q_{\lambda,\alpha}\|_{L^2} \leq \frac{CM^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}, \quad E[\Pi_M Q_{\lambda,\alpha}] \leq \frac{CM^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}}.$$

4.2. **Non-normalizable when $K > \|Q\|_{L^2(\mathbb{R})}$.** Pick $(1+2\alpha)\|Q\|_{L^2} < K_1 < K$, we have for any $\sigma \in (0,1)$, there exists $C_\sigma > 0$, such that

$$\begin{aligned}\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{\frac{1}{6}\|W_N\|_{L^6}^6}\mathbf{1}_{\|W_N\|_{L^2}\leq K}\big] \geq &\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{\frac{1}{6}\|W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M^{\perp}W_N\|_{L^2}<K_1}\cdot\mathbf{1}_{\|\Pi_M W_N\|_{L^2}\leq(K^2-K_1^2)^{1/2}}\big]\\ \geq &\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M^{\perp}W_N\|_{L^2}\leq K_1}\cdot\mathrm{e}^{-C_\sigma\|\Pi_M W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M W_N\|_{L^2}\leq(K^2-K_1^2)^{1/2}}\big],\end{aligned}$$

where $1 \ll M \ll N$ will be fixed later. Using the fact that $\Pi_M^{\perp} W_N$ is *independent* of $\Pi_M W_N$, the above quantity equals to

$$\epsilon_{M,N}\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M^{\perp}W_N\|_{L^2}\leq K_1}\big],$$

where

$$\epsilon_{M,N} := \mathbb{E}^{\mathbf{P}}[\mathrm{e}^{-C_\sigma\|\Pi_M W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M W_N\|_{L^2}\leq(K^2-K_1^2)^{1/2}}].$$

Note that for fixed M, $\epsilon_{M,N}$ is uniformly bounded from below in N.

Pick $0<\delta<K_1-\|Q_{\lambda,\alpha}\|_{L^2}$, then

$$\begin{aligned}
&\mathbb{E}^{\mathbf{P}}\big[\mathrm{e}^{\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N\|_{L^6}^6}\mathbf{1}_{\|\Pi_M^{\perp}W_N\|_{L^2}\le K_1}\big]\\
=&\mathbb{E}^{\mathbf{P}}\Big[\exp\Big(\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N\|_{L^6}^6\mathbf{1}_{\|\Pi_M^{\perp}W_N\|_{L^2}\le K_1}\Big)\Big]-\mathbf{P}[\|\Pi_M^{\perp}W_N\|_{L^2}>K]\\
=&I_{M,N}-\mathbf{P}[\|\Pi_M^{\perp}W_N\|_{L^2}>K],
\end{aligned}$$

where

$$I_{M,N}:=\mathbb{E}^{\mathbf{P}}\Big[\exp\Big(\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N\|_{L^6}^6\cdot\mathbf{1}_{\|\Pi_M^{\perp}W_N-Q_{\lambda,\alpha}\|_{L^2}\le\delta}\Big)\Big].$$

It suffices to show that for appropriately chosen M, $I_{M,N}$ is unbounded in N.

To this end, we apply the variational formula (3.2) to

$$F(W):=\frac{1-\sigma}{6}\|\Pi_M^{\perp}\Pi_N W\|_{L^6}^6\mathbf{1}_{\|\Pi_M^{\perp}W_N-Q_{\lambda,\alpha}\|_{L^2}\le\delta},$$

it suffices to show that

$$\mathbb{E}^{\mathbf{P}}\Big[\frac{1-\sigma}{6}\|\Pi_M^{\perp}W_N+Q_{\lambda,\alpha}\|_{L^6}^6\mathbf{1}_{\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}\le\delta}-\frac{1}{2}\|Q_{\lambda,\alpha}\|_{H^1}^2\Big]$$

is unbounded as $N\to\infty$.

As in the analysis in the previous subsection, we first minorize

$$\|W_N+Q_{\lambda,\alpha}\|_{L^6}^6\ge(1-\sigma)\|Q_{\lambda,\alpha}\|_{L^6}^6-C_\sigma\|W_N\|_{L^6}^6$$

for small enough $\sigma\in(0,1)$ to be specified later. As the expectation of $\|W_N\|_{L^6}^6$ is bounded, it suffices to show that

unbounded1 (4.3)
$$J_{M,N}:=\mathbb{E}^{\mathbf{P}}\Big[\frac{(1-\sigma)}{6}\|Q_{\lambda,\alpha}\|_{L^6}^6\mathbf{1}_{\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}\le\delta}-\frac{1}{2}\|Q_{\lambda,\alpha}\|_{H^1}^2\Big]$$

is unbounded from above as $N\to\infty$.

We minorize

$$\begin{aligned}
J_{M,N}\ge&\mathbb{E}^{\mathbf{P}}\Big[\Big(\frac{(1-\sigma)^2}{6}\|Q_{\lambda,\alpha}\|_{L^6}^6-\frac{1}{2}\|Q_{\lambda,\alpha}\|_{H^1}^2\Big)\mathbf{1}_{\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}\le\delta}\Big]\\
&-\frac{1}{2}\mathbb{E}^{\mathbf{P}}\big[\|Q_{\lambda,\alpha}\|_{H^1}^2\mathbf{1}_{\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}>\delta}\big].
\end{aligned}$$

For σ small enough,

$$\frac{(1-\sigma)^2}{6}\|Q_{\lambda,\alpha}\|_{L^6}^6-\frac{1}{2}\|Q_{\lambda,\alpha}\|_{H^1}^2\ge c_1(\alpha)\lambda^2\|Q\|_{L^6(\mathbb{R})}^6,$$

thus

$$\begin{aligned}
J_{M,N}\ge&c(\alpha)\lambda^2\|Q\|_{L^6}^6\mathbf{P}[\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}\le\delta]\\
&-c_1(\alpha)\lambda^2\|Q\|_{H^1}^2\mathbf{P}[\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}>\delta].
\end{aligned}$$

Hence it suffices to show that there exists $M\gg 1$ such that for any sufficiently large $N,\lambda=\frac{N}{10}$,

probabound (4.4)
$$\mathbf{P}[\|\Pi_M^{\perp}W_N-\Pi_M Q_{\lambda,\alpha}\|_{L^2}>\delta]<\frac{c(\alpha)}{4c_1(\alpha)}\frac{\|Q\|_{L^6}^6}{\|Q\|_{H^1}^2}.$$

Indeed, recall that $\|\Pi_M Q_{\lambda,\alpha}\|_{L^2} \le C\sqrt{\frac{M}{N}}$. As far as $M \ll \delta^2 N$,

$$\begin{aligned}\mathbf{P}[\|\Pi_M^\perp W_N - \Pi_M Q_{\lambda,\alpha}\|_{L^2} > \delta] \le& \mathbf{P}[\|\Pi_M^\perp W_N\|_{L^2} > \delta - \|\Pi_M Q_{\lambda,\alpha}\|_{L^2}] \\ \le& \mathbf{P}\big[\|\Pi_M^\perp W_N\|_{L^2} > \frac{\delta}{2}\big].\end{aligned}$$

From (3.9), we have (with $\delta_0 = \frac{\delta}{2}$)

$$\mathbf{P}\big[\|\Pi_M^\perp W_N\|_{L^2} > \frac{\delta}{2}\big] \le C_0 \mathrm{e}^{-cM\delta^2}.$$

Therefore, by choosing M large enough such that

$$C_0 \mathrm{e}^{-cM\delta^2} < \frac{c(\alpha)}{4c_1(\alpha)} \frac{\|Q\|_{L^6}^6}{\|Q\|_{H^1}^2},$$

then for any $N \gg \frac{M}{\delta^2}$, we obtain (4.4).

In summary, we have shown that $J_{M,N} \to \infty$ as $N \to \infty$, and this proves the non-normalizable for the case $K > \|Q\|_{L^2(\mathbb{R})}$.

•Final remark for the case $K = \|Q\|_{L^2(\mathbb{R})}$: For the normalizability, the answer is yes, thanks to the celebrated work of *Oh-Sosoe-Tolomeo* [5], where the Gibbs measure with mass truncation $\mathbf{1}_{[0,\|Q\|_{L^2(\mathbb{R})}]}(\|u\|_{L^2})$ is constructed. As we already know that for the mass-critical focusing NLS on $\mathbb{R}^d$ (which is the case for the quintic 1D NLS), at the mass threshold of the soliton, solutions can blowup. Moreover, *Merle* has shown that ([4]) any minimal mass blowup solution is the soliton modulo the pseudo-conformal symmetry. Thus the result in [5] illustrates that minimal mass blowup solutions are negligible from the macroscopic point of view.

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