SECOND MICROLOCALIZATION AND OBSERVABILITY

CHENMIN SUN

Abstract. In this note, we briefly recall the key ideas to perform the second microlocalization à la Anantharaman-Macià to prove the observation inequality of the Schrödinger equation on torus. This is an example of performing the second microlocalization along a coisotropic submanifold.

1. Second microlocalation à la Anantharaman-Macià

Consider the free Schrödinger evolution equation on $\mathbb{T}^2$:

$$i\partial_t u + \Delta u = 0.$$

Assume that $\omega = (a,b)_x \times \mathbb{T}_y$ and we want to prove the following observability

$$\|u(0)\|^2_{L^2(\mathbb{T}^2)} \leq C_T \int_0^T \|u(t)\|^2_{L^2(\omega)} dt \tag{1.1}$$

for any $T > 0$. From the standard strategy, we only need to prove (1.1) for the spectral localized data, hence we assume that

$$u(0) = \psi_h = \chi((h^2\Delta - 1)/\delta_h),$$

where $0 < h \ll \delta_h \ll 1$. We prove by contradiction. Assume that μ is a semi-classical measure of a subsequence of $(\psi_h)_{h>0}$. Then from the first propagation law (along the geodesic) and after making change of coordinate, we may assume that μ must be concentrated on the direction $\Xi_0 = (0,1)$, parallel to the control region ω. To obtain a contradiction, we perform a finer micro-localization following [AM10], called the second microlocalization along the coisotropic[1] submanifold $\Lambda^\perp := \{(x,y;\xi,\eta) : \xi = 0\}$. Note that $\mathbb{T}^2 \times (\Lambda^\perp \setminus \{0\})$ is a periodic orbit.

The idea is as follows: We know that $\mu = \mu \mathbf{1}_{\Lambda^\perp}$ is invariant along the direction Ξ_0, hence we may test the distributions by operators that are constant along $\Lambda^\perp$, or equivalently, with only Fourier coefficient in $\Lambda = \mathbb{R}\cdot(1,0)$. We need to deal with two asymptotic regimes: the transversal high frequency regime and the transversal low frequency regime. For the transversal high frequency regime, the associated Wigner distributions converge to a measure (second semiclassical measure) and it is invariant along the classical transversal flow. For the transversal low frequency regime, the associated Wigner distributions converge only to an operator-valued

[1]Recall that the coisotropic vector space of W of a symplectic vector space V (with a symplectic form σ) satisfies that $W^\sigma \subset W$.

measure, and it is invariant under the transversal quantum flow. Note that in the specific coordinate system, the transversal codirection is $(\xi, 0)$.

To test the transversal high frequencies, we use the quantization

$$a(x,\xi,\eta,\zeta)\mapsto \mathrm{Op}_h^{\epsilon}(a):=\mathrm{Op}\Big(a(x,h\xi,h\eta,\xi)(1-\chi(\epsilon\xi))\Big).$$

This quantization satisfies a good symbolic calculus (thanks to the uncertainty principal):

- Uniformly (in h, ϵ) boundness on L^2;
- Good symbolic calculus for compositions: gain of $\mathcal{O}(\epsilon^2)$ in the operator norm;
- Garding inequality: almost positivity.

These properties implies the existence of the second semi-classical measure $\widetilde{\mu}^{\Lambda}$ on $T^*\mathbb{T}^2\times\mathbb{S}_{\Lambda}$ ($\mathbb{S}_{\Lambda}$ consists of only two points in this special case), defined by the double limit (up to a subsequence)

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\big(\mathrm{Op}_h^{\epsilon}(a)\psi_h,\psi_h\big)_{L^2}=\langle\widetilde{\mu}^{\Lambda},a\rangle:=\int_{T^*\mathbb{T}^2\times\mathbb{S}_{\Lambda}}a(x,0,\eta,\zeta)\widetilde{\mu}^{\Lambda}(dxd\eta d\zeta).$$

For the transversal low frequencies, we use instead

$$a(x,\xi,\eta,\zeta)\mapsto \mathrm{Op}_{h,\epsilon}(a):=\mathrm{Op}\Big(a(x,h\xi,h\eta,\xi)\chi(\epsilon\xi)\Big).$$

This quantization does not satisfy a good symbolic calculus. It only defines a finite quantity (up to a subsequence)

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\big(\mathrm{Op}_{h,\epsilon}(a)\psi_h,\psi_h\big)_{L^2}=\langle\widetilde{\mu}_{\Lambda},a\rangle,$$

for every symbol a (in the class that we do not specify here). The object $\widetilde{\mu}_{\Lambda}$ is not a measure !

If we integrate $\widetilde{\mu}_{\Lambda}$ and $\widetilde{\mu}^{\Lambda}$ along the added fiber $d\zeta$, we obtain two positive measures $\mu_{\Lambda},\mu^{\Lambda}$ on $T^*\mathbb{T}^2$, with respectively. One verifies that the identity

$$\mu|_{\mathbb{T}^2\times\Lambda^{\perp}}=\mu^{\Lambda}+\mu_{\Lambda},$$

which means that we recover the total mass on $T^*\mathbb{T}^2\times\Lambda^{\perp}$ from μ_{Λ} and μ^{Λ}.

1.1. **Analysis of the transversal high frequency part.** In what follows, we denote by $\widetilde{\mu}^{\Lambda}(t)$, the second semi-classical measure (transversal high frequency part) associated with the solution

$$\phi_h(t)=S(t)\psi_h:=\mathrm{e}^{it\Delta}\psi_h$$

we will prove the transversal propagation law

Proposition 1.1. *Define the transversal flow*

$$s\mapsto\phi_s^1(x,y,\xi,\eta,\zeta):=\Big(x+s\frac{\zeta}{|\zeta|},y,\xi,\eta,\zeta\Big),$$

then for a.e.$t\in\mathbb{R}$, we have

$$(\phi_s^1)_*\widetilde{\mu}^{\Lambda}(t,\cdot)=\widetilde{\mu}^{\Lambda}(t,\cdot).$$

Proof. Using the equation of $\psi_h(t)$, we have

$$\begin{aligned}\frac{d}{dt}\big(\mathrm{Op}_h^\epsilon(b)\psi_h(t),\psi_h(t)\big)_{L^2} =& i\big([\mathrm{Op}_h^\epsilon(b),\Delta]\psi_h(t),\psi_h(t)\big)_{L^2} \\ =& i\big(\big(\mathrm{Op}_h^\epsilon(\partial_x b)\partial_x+\partial_x\mathrm{Op}_h^\epsilon(b)\big)\psi_h(t),\psi_h(t)\big).\end{aligned} \tag{1.2}$$

Let $b(x,\xi,\eta,\zeta)=\frac{1}{|\zeta|}a(x,\xi,\eta,\zeta)(1-\chi(\epsilon\zeta))$, then the principal part on the right side is

$$2\Big(\mathrm{Op}_h^\epsilon\big(\frac{\zeta}{|\zeta|}\cdot\partial_x a\big)\psi_h(t),\psi_h(t)\Big)_{L^2}.$$

Let $\varphi(t)\in C_c^1(\mathbb{R}_t)$ be a test function. Multiplying $\varphi(t)$ on both sides of (1.2) and integrating over $t\in\mathbb{R}$, we obtain that

$$-\int_{\mathbb{R}}\varphi'(t)\big(\mathrm{Op}_h^\epsilon(b)\psi_h(t),\psi_h(t)\big)_{L^2}dt=2\int_{\mathbb{R}}\varphi(t)\big(\mathrm{Op}_h\big(\frac{\zeta}{|\zeta|}\cdot\partial_x a\big)\psi_h(t),\psi_h(t)\big)_{L^2}+o(1). \tag{1.3}$$

Note that $|\zeta|\geq\frac{1}{\epsilon}$ on the support of $1-\chi(\epsilon\zeta)$, by Caldrón-Vaillancourt, the left side of (1.3) tends to 0 when we take the double limit $h\to0,\epsilon\to0$. This completes the proof. □

1.2. **Analysis of the transversal low frequency part.** Denote by $H=L^2(\Lambda)$, the L^2 space of functions depending only on x variable. The for fixed ϵ, operator $\mathrm{Op}_{h,\epsilon}(a)$ can be viewed as h-pseudo-differential operators taking values in $\mathcal{K}(H)$, the space of compact operators on H. We write

$$\big(\mathrm{Op}_{h,\epsilon}(a)\psi_h,\psi_h\big)_{L^2}=\big(a_\epsilon(hD_y,\cdot))\psi_h,\psi_h\big)_{L^2},$$

where the notation $a_\epsilon(hD_y,\cdot)=a(x,hD_x,hD_y,D_x)\chi(\epsilon D_x)$ emphasizes this viewpoint. Denote by $\mathcal{L}^1(H)$ the space of trace class operators on H and $\mathcal{B}(H)$ the space of bounded operators. It turns out that there exists a $\mathcal{L}^1(H)$ valued positive Hermitian measure $\widetilde{\rho}_\Lambda(t,\cdot)\in\mathcal{M}_+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(H))$, such that, for any $a=a(x,\xi,\eta,\zeta)$, independent of the variable y, we have

$$\langle\widetilde{\mu}_\Lambda(t),a\rangle=\mathrm{Tr}_H\int_{T^*\mathbb{T}_\Lambda^\perp}a(\eta,\cdot)\widetilde{\rho}_\Lambda(t,dy,d\eta),$$

where $a(\eta,\cdot)$ is a symbol on $T^*\mathbb{T}_{\Lambda^\perp}$, taking values in $\mathcal{B}(H)$ (see Appendix for detailed discussions). From [PG90], with $\nu=\mathrm{Tr}(\widetilde{\rho}_\Lambda)$, there exists $f_\Lambda(t,y,\eta)\in L^1(T^*\Lambda_\Lambda^\perp;\mathcal{L}^1(H),d\nu)$, such that

$$\widetilde{\rho}_\Lambda(t,dy,d\eta)=f_\Lambda(t,y,\eta)\nu(t,dy,d\eta).$$

Hence

$$\int_{T^*\mathbb{T}^2\times\Lambda}a(x,y,\xi,\eta)\widetilde{\mu}_\Lambda(t,dx,d\xi,d\eta,d\zeta)=\mathrm{Tr}_H\int_{T^*\mathbb{T}_\Lambda^\perp}a(\eta,\cdot)f_\Lambda(t,y,\eta)\nu(t,dy,d\eta). \tag{1.4}$$

Now we want to find the link of the measure $\widetilde{\rho}_\Lambda(t)$ with $\widetilde{\rho}_\Lambda(0)$.

Doing the same manipulation as in (1.2), we have,

$$\frac{d}{dt}\big(a_\epsilon(hD_y,\cdot)\psi_h(t),\psi_h(t)\big)_{L^2}=i\big([a_\epsilon(hD_y,\cdot),\Delta_\Lambda]\psi_h(t),\psi_h(t)\big)_{L^2},$$

and $\Delta_\Lambda = \partial_x^2$ in our situation. Passing to the limit formally, we obtain that

$$\frac{d}{dt}\mathrm{Tr}_H \int_{T^*\mathbb{T}_{\Lambda^\perp}} a(\eta,\cdot)d\widetilde{\rho}_\Lambda = i\,\mathrm{Tr}_H \int_{T^*\mathbb{T}_{\Lambda^\perp}} [a(\eta,\cdot),\Delta_\Lambda]d\widetilde{\rho}_\Lambda.$$

Using $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we derive the Schrödinger equation for the trace-class valued measure:

$$\frac{d}{dt}\widetilde{\rho}_\Lambda = i[\Delta_\Lambda, \widetilde{\rho}_\Lambda]. \tag{1.5}$$

Note that this is exactly the Schrödinger equation for the density operator in Heisenberg's picture (quantum statistics).

Take $(e_k)_{k\in\mathbb{Z}}$, a canonical orthonomal basis of $H = L^2(\Lambda)$ (for this special Λ, we can take $e_k(x) = \mathrm{e}^{ikx}$). Without loss of generality, we may assume that $\widetilde{\rho}_\Lambda(0)$ is diagonalized under this basis. Note that from the first propagation law, $\widetilde{\rho}_\Lambda(t,\cdot)$ is constant along $\Lambda^\perp$ (thus is constant in y). Hence the equation (1.5) simply means that

$$\widetilde{\rho}_\Lambda(t.\cdot) = S_\Lambda(-t)\widetilde{\rho}_\Lambda(0,\cdot)S_\Lambda(t), \tag{1.6}$$

where $S(t) = S_\Lambda(t)\otimes S_{\Lambda^\perp}(t)$ and in the special situation $S_\Lambda(t) = \mathrm{e}^{it\partial_x^2}$. This implies that $\widetilde{\rho}_\Lambda(t\cdot)$ is also diagonalized under the basis of $(e_k)_{k\in\mathbb{Z}}$.

1.3. **Proof of the observability.** The contradiction assumption leads to

$$\int_0^T \langle \mu(t,\cdot), \mathbf{1}_{\omega\times\Lambda^\perp}\rangle dt = 0.$$

In particular,

$$\int_0^T \langle \mu^\Lambda(t,\cdot), \mathbf{1}_{\omega\times\Lambda^\perp}\rangle dt = 0, \quad \int_0^T \langle \mu_\Lambda(t,\cdot), \mathbf{1}_{\omega\times\Lambda^\perp}\rangle dt = 0.$$

Thus, by invariance of $\widetilde{\mu}^\Lambda$ (hence μ^Λ) under the transversal flow ϕ_s^1, we have

$$\int_0^T dt \quad \langle \mu^\Lambda(t,\cdot), \mathbf{1}_{\mathbb{T}^2\times\Lambda^\perp}\rangle dt = 0.$$

Since $\mathbf{1}_\omega(x,y) = \mathbf{1}_{(a,b)}(x)$ does not depend on y, we have from(1.4) that

$$\int_0^T dt \int_{T^*\mathbb{T}^2} \mathbf{1}_{(a,b)}(x)\mu_\Lambda(t,dxd\xi d\eta) = \int_0^T dt\mathrm{Tr}_H \int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathbf{1}_{(a,b)}\widetilde{\rho}_\Lambda(t,d\eta).$$

Write

$$\widetilde{\rho}_\Lambda(0,d\eta) = \sum_{k\in\mathbb{Z}} \widetilde{\rho}_k(d\eta)(e_k,\cdot)_H e_k,$$

from (1.6), we have

$$\begin{aligned}\int_0^T dt\mathrm{Tr}_H \int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathbf{1}_{(a,b)}\widetilde{\rho}_\Lambda(t,d\eta) &= \int_0^T dt \int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Tr}_H\big(S_\Lambda(t)\mathbf{1}_{(a,b)}S_\Lambda(-t)\widetilde{\rho}_\Lambda(0,d\eta)\big) \\ &= \sum_k \int_0^T dt \int_{T^*\mathbb{T}_{\Lambda^\perp}} \big(\big(S_\Lambda(t)\mathbf{1}_{(a,b)}S_\Lambda(-t)\big)e_k, e_k\big)_H \widetilde{\rho}_k(d\eta).\end{aligned}$$

From 1D observability, the last term on the right side is bounded from below by

$$c\int_0^T dt \int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Tr}_H(\widetilde{\rho}_\Lambda(t, d\eta)) = c\mu_\Lambda(T^*\mathbb{T}^d).$$

Hence $\mu_\Lambda = 0$ and this is a contradiction.

Appendix A. Operator-valued semiclassical measures

A.1. **Trace-class operators.** In this appendix, we loosely follow [PG90] to define the operator-valued semi-classical measure, needed in Subsection 1.2.

First we recall some notations and facts. Let $\mathcal{H}$ be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ the spaces of bounded linear operators, compact operators on $\mathcal{H}$, with respectively. Recall that $\mathcal{K}(\mathcal{H})$ is the closure of the finite-rank operators with respect to the topology of $\mathcal{L}(\mathcal{H})$. Denote by $\mathcal{L}^1(\mathcal{H})$ the space of *trace-class* operators on $\mathcal{H}$, that is bounded operators A on $\mathcal{H}$ such that

$$\|A\|_1 := \sum_{k=1}^{\infty} \langle (AA^*)^{\frac{1}{2}} e_k, e_k\rangle_{\mathcal{H}} < \infty$$

for an orthonormal base $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$. In this case, we define the *trace* of A by

$$\mathrm{Tr}(A) := \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle_{\mathcal{H}}$$

As an exercise, one can verify that the above definition is *independent* of the choice of the orthonormal base.

Another relevant subspace is the space of *Hilbert-Schmidt* operators $\mathcal{L}^2(\mathcal{H})$, consisting of bounded linear operators A such that

$$\|A\|_2 := (\mathrm{Tr}(AA^*))^{\frac{1}{2}} < \infty.$$

The following proposition collects some basic facts about the trace-class operators, the Hilbert-Schmidt operators and the compact operators:

Proposition A.1. (1) $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^2(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$.

(2) *$\mathcal{L}^1(\mathcal{H})$ is an ideal (left and right) of the Banach algebra $\mathcal{L}(\mathcal{H})$. More precisely, if $S \in \mathcal{L}(\mathcal{H}), T \in \mathcal{L}^1(\mathcal{H})$, then both ST and TS are also in $\mathcal{L}^1(\mathcal{H})$ and* $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$.

(3) *A bounded linear operator T on $L^2(X,\mu)$ for some measure space (X,μ) is Hilbert-Schmidt if and only if $T = T_K$ for some integral kernel $K \in L^2(X\times X)$. Furthermore,*

$$\|T_K\|_2^2 = \int_{X\times X} |K(x,y)|^2 \mu(dx)\mu(dy).$$

(4) *Let T_K be a bounded linear operator on $L^2(X,\mu)$ for some measure space (X,μ) with integral kernel K. If $T_K \in \mathcal{L}^1(L^2(X,\mu))$, then*

$$\mathrm{Tr}(T_K) = \int_X K(x,x)\mu(dx).$$

One importance of the trace-class operators is because it is the dual of the space of compact operators:

Proposition A.2. *The dual $\mathcal{K}'(\mathcal{H})$ of the space of compact operators $\mathcal{K}(\mathcal{H})$ is $\mathcal{L}^1(\mathcal{H})$, with the duality bracket $\langle A, K\rangle = \mathrm{Tr}(KA)$.*

A.2. **Operator-valued semiclassical measures.** Recall that in Subsection 1.2, we use

$$\mathrm{Op}_{h,\epsilon}(a) = a(x,y,hD_x,hD_y,D_x)\chi(\epsilon D_x)$$

to capture the information of the transversal low frequency portions. Now we are going to rigorously justify the existence of the operator-valued semiclassical defect measure. Recall the notation $\mathcal{H} = L^2(\mathbb{T}_x)$ and $\mathcal{M}_+(T^*\mathbb{T}_y;\mathcal{L}^1(\mathcal{H}))$ denote the space of positive Hermitian measures valued in $\mathcal{L}^1(\mathcal{H})$. For simplicity, we do note consider the time-dependence, as in the context of Subsection 1.2:

Theorem 1 (P. Gérard)**.** *Let $(u_h)_{h>0}$ be a bounded sequence in $L^2(\mathbb{T}^2)$ such that $\|u_h\|_{L^2(\mathbb{T}^2)} = 1$. After extracting to a subsequence of $h \to 0$ and $\epsilon \to 0$, there exists $\rho(dyd\eta) \in \mathcal{M}_+(T^*\mathbb{T}_y;\mathcal{L}^1(\mathcal{H}))$, such that for any compactly supported symbol $b(x,\xi,\eta,\sigma)$, independent of y, we have*

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\langle \mathrm{Op}_{h,\epsilon}(b)u_h, u_h\rangle_{L^2} = \mathrm{Tr}_{\mathcal{H}}\Big[\int_{T^*\mathbb{T}_y}\widetilde{b}(\eta,\cdot)\rho(dyd\eta)\Big], \tag{A.1}$$

where $\widetilde{b}(y,\eta,\cdot) = b(x,y,0,\eta,D_x)$, regarded as a symbol in y,η, taking values in $\mathcal{K}(L^2_x)$.

Proof. The proof consists of two steps. In the first step, we chose special operator-valued symbols of the form $b(y,\eta)K$, where $b(y,\eta)$ are scaler symbols and $K \in \mathcal{K}(L^2_x)$, independent of y,η. This allows us to construct the operator-valued measure $\rho(dyd\eta)$. Then in the second step, we approximate a general $\mathcal{K}(\mathcal{H})$-valued symbol by finite-rank approximation.

•Step 1: Measure construction

Let D be a countable subset of $C_c^\infty(T^*\mathbb{T}_y)$ which is dense in $C_0(T^*\mathbb{T}_y)$. For any $b \in D$, we define the linear form on $\mathcal{K}(\mathcal{H})$ by

$$\mathcal{L}_{h,\epsilon}(b,K) := \langle \mathrm{Op}_h(b)Ku_h, u_h\rangle_{L^2}.$$

Note that

$$|\mathcal{L}_{h,\epsilon}(b,K)| \leq \|\mathrm{Op}_h(b)\|_{\mathcal{L}(L^2_y)}\|K\|_{\mathcal{L}(\mathcal{H})}.$$

Thus $\mathcal{L}_{h,\epsilon}(b,\cdot)$ is equicontinuous in the weak *-topology of $\mathcal{L}(\mathcal{H})$. By the Banach-Alaoglu theorem, there exist a subsequence and an element $\rho_0(b) \in \mathcal{L}^1(\mathcal{H})$, such that for all $K \in \mathcal{K}(\mathcal{H})$,

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathcal{L}_{h,\epsilon}(b,K) = \mathrm{Tr}(K\rho_0(b)). \tag{A.2}$$

Since D is countable, by the diagonal extracting, we may assume that, with the same subsequence, (A.2) holds for every $b \in D$. Denote by $\mathcal{L}(b, K) = \mathrm{Tr}(K\rho_0(b))$.

By the Caldrón-Vaillancourt theorem, we have

$$|\mathrm{Tr}(K\rho_0(b))| \leq C\|b\|_{L^\infty}\|K\|_{\mathcal{L}(\mathcal{H})}.$$

This shows that $\rho_0 : D \to \mathcal{L}^1(\mathcal{H})$ is a bounded linear functional on D. Since D is dense in $C_0(T^*\mathbb{T}_y)$, the linear map ρ_0 can be extended to $C_0(T^*\mathbb{T}_y)$. Therefore, by the Riesz theorem, there exists a $\mathcal{L}^1(\mathcal{H})$-valued Radon measure ρ on $T^*\mathbb{T}_y$, such that $\rho_0(b) = b(y,\eta)\rho(dyd\eta)$, hence

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathcal{L}_{h,\epsilon}(b, K) = \mathrm{Tr}_{\mathcal{H}}\Big[\int_{T^*\mathbb{T}_y} b(y,\eta)K\rho(dyd\eta)\Big].$$

By linearity, the above identity can be extended to linear combinations of the form $\sum_{j=1}^N b_j(y,\eta)K_j$.

Next we show that $\rho_0 \in \mathcal{M}_+(T^*\mathbb{T}_y; \mathcal{L}^1(\mathcal{H}))$, i.e. for any non-negative function $\varphi \in C_0(T^*\mathbb{T}_y)$, $\langle \rho_0, \varphi\rangle \in \mathcal{L}^1(\mathcal{H})$ is a non-negative Hermitian operator of $\mathcal{H}$. To this end, we mimic the proof of Lemma 1.2 [PG90] to get

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathcal{L}_{h,\epsilon}(\varphi, K) \geq 0,$$

for any non-negative function $\varphi(y,\eta) \in C_0^(T^*\mathbb{T}_y)$ and non-negative compact operator $K \in \mathcal{K}(\mathcal{H})$. In particular, $\mathrm{Tr}_{\mathcal{H}}[\rho_0(b)K] \geq 0$. By taking $K = \pi_j$, where π_j is the projector to the eigenspace of $\varphi(\varphi)$ of eigenvalue $\lambda_j(\varphi)$, we obtain that $\lambda_j(\varphi) \geq 0$. This shows that $\rho_0(\varphi) \geq 0$, hence $\rho_0 \in \mathcal{M}_+(T^*\mathbb{T}_y; \mathcal{L}^1(\mathcal{H}))$.

•Step 2: Extending to general operator-valued symbols

For a general symbol $b(x,\xi,\eta,\sigma)$, to verify (A.1), the basic idea is to use the compactness to approximate a by linear combinations of the form $\sum_{j=1}^N b_j(y,\eta)K_j$ for which (A.1) hold. To this end, we pick an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ and compute formally the inner product

$$\langle \mathrm{Op}_{h,\epsilon}(b)u_h, u_h\rangle_{L^2_{x,y}}.$$

We expand

$$u_h(x,y) = \sum_{j\in\mathbb{N}} u_h^j(y)e_j(x),$$

where $u_h^j(y) = \langle u_h, e_j\rangle_{L^2_x}$. Denote by $\widetilde{a}(\eta,\cdot)$, the Then

$$\langle \mathrm{Op}_{h,\epsilon}(b)u_h, u_h\rangle_{L^2} = \sum_{j,j'\in\mathbb{N}} \langle \widetilde{b}_\epsilon^{j,j'}(hD_y)u_h^j, u_h^{j'}\rangle_{L^2_y}, \tag{A.3}$$

where $\widetilde{b}_\epsilon^{j,j'}(hD_y)$ is the quantization of the symbol $\widetilde{b}_\epsilon^{j,j'}(\eta)$ defined via

$$\widetilde{b}_\epsilon^{j,j'}(\eta) := \langle \widetilde{b}_\epsilon(\eta,\cdot)e_j, e_{j'}\rangle_{L^2_x} = \langle b(x, hD_x, \eta, D_x)\chi_\epsilon(D_x)e_j, e_{j'}\rangle_{L^2_x}.$$

Since $\langle \widetilde{b}_\epsilon^{j,j'}(hD_y)u_h^j, u_h^{j'}\rangle_{L^2_y} = \langle \widetilde{b}_\epsilon^{j,j'}(hD_y)E_{j,j'}u_h, u_h\rangle_{L^2_{x,y}}$, where

$$E_{j,j'}f = \langle f, e_{j'}\rangle_{L^2_x}e_j$$

is of rank 1. Formally passing to the limit in (A.3), we get

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\langle \mathrm{Op}_{h,\epsilon}(b)u_h,u_h\rangle_{L^2}=\sum_{j,j'\in\mathbb{N}}\mathrm{Tr}_{\mathcal{H}}\Big[\int_{T^*\mathbb{T}_y}\widetilde{b}_{\epsilon}^{j,j'}(\eta)E_{j,j'}\rho(dyd\eta)\Big]=\mathrm{Tr}_{\mathcal{H}}\Big[\int_{T^*\mathbb{T}_y}\widetilde{b}(\eta,\cdot)\rho(dyd\eta)\Big].$$

In order to make the above limiting procedure rigorous, it suffices to show that

$$\lim_{N\to\infty}\big\langle\big(\pi_N\mathrm{Op}_{h,\epsilon}(b)\pi_N-\mathrm{Op}_{h,\epsilon}(b)\big)u_h,u_h\big\rangle_{L^2_{x,y}}=0, \tag{A.4}$$

uniformly in $\epsilon,h\in(0,1)$, where π_N is the orthogonal projection onto the space generated by $(e_j)_{j\leq N}$. First, since $b(\eta,x,\xi,\eta)$ is compactly supported, hence for each $\eta\in\mathbb{R}$,

$$\lim_{N\to\infty}\|\widetilde{b}(\eta,\cdot)-\pi_N\widetilde{b}(\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)}=0, \tag{A.5}$$

where $\widetilde{b}_{\epsilon}(\eta,\cdot)=b(\eta,x,hD_x,D_x)\chi_{\epsilon}(D_x)$. Moreover, by the uniform boundedness

$$\sup_{\eta\in\mathbb{R},N}\|\partial_\eta^\alpha\widetilde{b}(\eta,\cdot)\|_{\mathcal{L}(L^2_x)}+\sup_{\eta\in\mathbb{R},N}\|\pi_N\partial_\eta^\alpha\widetilde{b}(\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)}\leq C_{\alpha,b},$$

we deduce that the sequence of functions $\Phi_N(\eta):=\|\widetilde{b}(\eta\cdot)-\pi_N\widetilde{b}(\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)}$ are equicontinuous and supported on a fixed compact set. This implies that (A.5) holds, uniformly in η. By the same argument,

$$\lim_{N\to\infty}\|\partial^\alpha\widetilde{b}(\eta,\cdot)-\pi_N\partial^\alpha\widetilde{b}(\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)}=0, \tag{A.6}$$

holds uniformly in η, for all α. By the Caldrón-Vaillancourt theorem (for the vector-valued symbol), we have

$$\begin{aligned}\big\langle\big(\pi_N\mathrm{Op}_{h,\epsilon}(b)\pi_N-\mathrm{Op}_{h,\epsilon}(b)\big)u_h,u_h\big\rangle_{L^2_{x,y}}\leq& C\sum_{|\alpha|\leq 10}\|\partial^\alpha\widetilde{b}(h\eta,\cdot)-\pi_N\partial^\alpha\widetilde{b}(h\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)}\|u_h\|_{L^2}^2\\ \leq& C\sum_{|\alpha|\leq 10}\sup_{\eta\in\mathbb{R}}\|\partial^\alpha\widetilde{b}(\eta,\cdot)-\pi_N\partial^\alpha\widetilde{b}(\eta,\cdot)\pi_N\|_{\mathcal{L}(L^2_x)},\end{aligned}$$

which converges to 0, uniformly in $h,\epsilon\in(0,1)$. This completes the proof of Theorem 1. □

References

[AM10] N. Anantharaman, F. Macià, *Semiclassical measures for the Schrödinger equations on the torus*, JEMS.

[BS19] N. Burq, C. Sun *Time optimal observability for the Grushin-Schrödinger equation*, preprint, arXiv:1910.03691.

[PG90] P. Gérard, *Microlocal defect measure*, Commu. PDEs, .

[Zw12] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, (2012) AMS.